1 Brief review of loss of interference from the Schrödinger perspective

We begin to motivate our approach by reviewing past attempts to analyze the disappearance of interference whenever it is possible to detect through which slit the particle passes. The original debate was famously conducted by Einstein and Bohr. Einstein attempted to challenge the consistency of quantum mechanics by arguing that a Which Way Measurement (WWM) could be performed without destroying the interference pattern by measuring the transverse recoil (i.e. the transverse momentum kick) of the double-slit screen after the particle passed through. Bohr maintained that the consistency of quantum mechanics depended on the destruction of the interference pattern when WWM information is obtained. He showed that the measurement-induced uncertainty created in the transverse position of the screen by an accurate measurement of the transverse momentum was sufficient to destroy the interference pattern.

This reasoning leads to a paradox which helps to motivate our approach. It has been argued (borrowing from the discussion of the "Heisenberg microscope") that if the particle were "observed" at the right slit, then the photon involved in this observation should have a wavelength $\lambda \leq D/2$ and a corresponding momentum uncertainty $\Delta p > 2\hbar/D$. This momentum uncertainty is imparted to the particle making its wave number $k = p/\hbar$ uncertain, thereby destroying the interference pattern.

This argument is incorrect. To see this, assume that a sensitive detector, placed at the left slit, failed to detect any particle. We then know that all particles passed through the right slit. The interference pattern will then be completely destroyed despite the fact that there was no interaction with the detector! [10] [7] One might suppose that since the action of opening/closing the left slit never caused an interaction with the particle at the right slit, then nothing associated with the particle should change. But, it was first pointed out by APP [10] that in this scenario when a WWM is performed without actually interacting with the interfering particle, then the probability distribution of the momenta *does* change, although none of the moments of the momenta change.

To best resolve this paradox, we need to take a step back. We note that the effect of a generic interaction or collision between any two quantum systems can be characterized by a change in the probability distribution of the momentum i.e. going from an initial probability distribution, $\rho_i(p)$, to a final distribution, $\rho_f(p)$. We can analyze this change in two ways[2]:

1. Look at moments such as $\langle p^n \rangle = \int \rho(p) p^n dp$ and calculate $\delta\langle p^n \rangle = \langle p^n \rangle_f - \langle p^n \rangle_i$, and thus ask how the interaction affected these averages. This is the usual approach.

2. Or, we may look at the fourier transform of the probability distribution $\int \rho(p) e^{\frac{i}{\hbar}pD} dp$. (We will later see that these functions, $\langle e^{\frac{i}{\hbar}pD} \rangle$, are precisely the observables that are sensitive to the relative phase.) To analyze the effect of the interaction, we calculate $\langle e^{\frac{i}{\hbar}pD} \rangle_f - \langle e^{\frac{i}{\hbar}pD} \rangle_i$ and ask how the interaction affected these averages.

In principle, one can discuss the effect of interactions using (1) or (2), since knowing (2) for all D is equivalent to knowing (1) for all n.

1.1 Analyzing changes in probability distribution using method 1: moments of the conserved quantity

Scully [7] et al and Storey [8] et al further debated the issues introduced by APP, resulting in many hundreds of cited papers.

Scully et al were dissatisfied with Bohr's original response to Einstein. They suggested that a microscopic pointer (i.e. a micro-maser) could be used in such a way that the interference in a WWM is destroyed without imparting any momentum to the particle (just as we alluded to earlier in the discussion of the case in which a sensitive detector failed to find the particle at the left slit).

However, Storey (et al) countered this, stating that the momentum distribution does change when WWMs are made. They noted that having a plane wave with initial $\Delta x = \infty$ and $\Delta p = 0$ impinge on the 2-slits projects the initial plane wave onto "lumps" which therefore have a significant Δp.

The principal components of both camps' arguments were previously put forward in APP, i.e. there is both a change in probability and no change in the moments. But, can we actually observe the change in the probability of the momentum when the left slit is open or closed? To determine whether the momentum is disturbed by the WWM, the momentum of the particle must be known before the WWM and after. However, if an ideal

[2] We consider momentum here, but our comments apply to any conserved quantity.

measurement is made of the momentum before the WWM, then we have effectively measured the interference, rendering useless the subsequent WWM.

The techniques of weak measurement have proven very useful in scenarios like this requiring manifestation of two opposing situations, i.e. to have a "have-your-cake-and-eat-it" solution. Weak measurements have had a direct impact on the central "mystery" alluded to by Feynman concerning indeterminism, namely the fact that the past does not completely determine the future. This mystery was accentuated by an assumed "time-asymmetry" within quantum mechanics, namely the assumption that measurements only have consequences **after** they are performed, i.e. towards the future. Nevertheless, a positive spin was placed on quantum mechanic's non-trivial relationship between initial and final conditions by Aharonov, Bergmann and Lebowitz (ABL) [4] who showed that the new information obtained from future measurements was also relevant for the **past** of quantum systems and not just the future. This inspired ABL to re-formulate quantum mechanics in terms of *pre- and post-selected ensembles*. The traditional paradigm for ensembles is to simply prepare systems in a particular state and thereafter subject them to a variety of experiments. These are "pre-selected-only-ensembles." For pre-**and-post-selected**-ensembles, we add one more step, a subsequent measurement or post-selection. By collecting only a subset of the outcomes for this later measurement, we see that the "pre-selected-only-ensemble" can be divided into sub-ensembles according to the results of this subsequent "post-selection-measurement." Because pre- and post-selected ensembles are the most refined quantum ensemble, they are of fundamental importance and have revealed novel aspects of quantum mechanics that were missed before, particularly the weak value which has been confirmed in numerous weak measurement experiments. Weak values have led to quantitative progress on many questions in the foundations of physics [19] including interference [12], field theory, in tunneling, in quantum information such as the quantum random walk, in foundational questions, in the discovery of new aspects of mathematics, such as Super-Fourier or super-oscillations, etc. It has also led to generalizations of quantum mechanics that were missed before.

While it is standard lore that the wave and particle nature cannot manifest at the same time, weak measurements on pre- and post-selected ensembles *can* provide information about both the (pre-selected) interference pattern and about the (post-selected) direction of motion for each particle. This aspect of weak measurements formed the basis for the first application of weak measurements to study the change in momentum for WWM within the double-slit setup as presented by Wiseman [9]. This was followed by an experiment(Mir, Lundeen, Mitchell, Steinberg, Garretson and Wiseman [23]). Besides clarifying the different definitions and different measurements (etc) used by both sides of the debate, Wiseman and Mir et al show that the momentum transfer can be observed for the spatial wavefunction used in the 2-slits (as opposed to momentum eigenstates) by using weak measurements.

They implemented the weak measurement with position shifts and polarization rotations in a large optical interferometer. Plotting the conditional probability to obtain a particular momentum (given the appropriate post-selection) and integrating over all possible post-selections, they were able to verify both the Scully and Storey viewpoints. With respect to Scully [7], they show that none of the moments of the momentum change. With respect to Storey [8], they show that the momentum does extend beyond a certain width.

However, there are inherent limitations to any approach based on analyzing changes in the probability for momenta through changes in the moments. For example, while momentum is of course conserved, there is no definite connection between the probability of an individual momentum before and after an exchange between the interfering particle and the slit. Furthermore, the analysis in terms of moments does not offer any intuition as to ***how*** or ***why*** the probability of momentum changes.

1.2 Analyzing changes in probability distributions using using method 2: fourier transform of the conserved quantity

When compared to the first (traditional) approach based on the moments, the second approach focusing on the fourier transform of the probability distribution has many advantages, both mathematical and physical. In this section, we briefly review some of the mathematical advantages, leaving much of the physical advantages to the rest of the article.

The first "moments" approach to interference derived from intuitions developed with wavefunctions consisting of just one "lump." In these cases, the averages of x (or of p) evolve according to *local* classical equations of motion. Also the uncertainties $(\Delta x)^2 \equiv \overline{(\hat{x})}^2 - \overline{\hat{x}}^2$ and $(\Delta p)^2 \equiv \overline{(\hat{p})}^2 - \overline{\hat{p}}^2$, describing the spread in these variables, have properties similar to those of the spread of variables in a classical situation with unsharply defined initial conditions and which evolve according to diffusion-like rules.

This drastically changes when we have two or more separate "lumps" of the wavefunction. Indeed, the wavefunction, after passing through the symmetric two-slits, consists of a superposition of two identical, but physically disjoint "lumps," $\psi_{\rm L}$ and $\psi_{\rm R}$ (see fig. 1):

$$|\Psi_\alpha\rangle = \frac{1}{\sqrt{2}}\{|\psi_{\rm L}\rangle + e^{i\alpha}|\psi_{\rm R}\rangle\} \tag{1}$$

Collapsing it to just $\psi_{\rm R}(x) \equiv \langle x|\psi_{\rm R}\rangle$ does *not* change Δp nor the expectation values of any finite order polynomial in p, as none of these local operators have a non-vanishing matrix element between the disjoint "lumps" of the wavefunction. In other words, measuring through which slit the particle passes does not have to increase the uncertainty in momentum. Later in this article we will review another uncertainty relationship which is more relevant for this issue.

Up until now we have focused on the disappearance of interference upon WWM. But the other fundamental mystery highlighted by Feynman remains: namely, how does a particle localized at the right slit "know" whether the left slit is open or closed? The first approach based on moments tell us nothing about this mystery. The decisive importance of the second "fourier transform" approach for this mystery is best illustrated through a basic theorem which characterizes all interference phenomenon: all **moments** of both position and momentum are *independent* of the relative phase parameter α (until the wavepackets overlap):

Theorem I: Let $\Psi_\alpha = \psi_{\rm L}(x,t) + e^{i\alpha}\psi_{\rm R}(x,t)$ such that there is no overlap of $\psi_{\rm L}(x,0)$ and $\psi_{\rm R}(x,0)$. If n and m are integers, then for all values of t, and choices of α, β:

$$\int [\Psi^*_\alpha(x,t)\Psi_\alpha(x,t) - \Psi^*_\beta(x,t)\Psi_\beta(x,t)]x^m p^n dx = 0 \tag{2}$$

For the particular double-slit wavefunction, it is easy to see that if there is no overlap between $\psi_{\rm L}$ and $\psi_{\rm R}$ then nothing of the form $\int_{-\infty}^{\infty} \Psi^* x^m p^n \Psi dx$ will depend on α for any value of m and n. Furthermore, expanding $\int\{\psi_{\rm L}+e^{-i\alpha}\psi_{\rm R}\}^* x^m p^n \{\psi_{\rm L}+e^{i\alpha}\psi_{\rm R}\}dx$, we see that only the cross terms, i.e. $\langle\psi_{\rm L}|x^m p^n|e^{i\alpha}\psi_{\rm R}\rangle$, have the *possibility* of depending on α; but operators of the form $x^m p^n$ cannot change the fact that $\psi_{\rm R}$ and $\psi_{\rm L}$ do not overlap. When integrated, these terms vanish and are therefore insensitive to the relative phase.

This suggests that these dynamical variables (e.g. $\langle x\rangle$, $\langle p\rangle$, Δx, Δp) are not the most appropriate to describe quantum interference phenomena. What observables, then, are sensitive to this interference information which appears to be stored in a subtle fashion? To fully capture the physics of these scenarios with wavefunctions composed of multiple lumps, non-polynomial and *non-local* operators, connecting the disjoint parts are required. For many, equi-distant slits, these are the discrete translation by $\pm D$, namely $\exp\{\pm\frac{i}{\hbar}\hat{p}D\}$, effecting $\exp\{-\frac{i}{\hbar}\hat{p}D\}\psi_{\rm R}(x) \to \psi_{\rm R}(x-D)$ which overlaps with $\psi_{\rm L}(x)$. The expectation value of the translation operator $\exp\{\frac{i}{\hbar}\hat{p}D\}$ ***does*** depend on α: $\langle\Psi_\alpha \mid \exp\{i\hat{p}D/h\} \mid \Psi_\alpha\rangle = e^{-i\alpha}/2$.

This provides the basis for a mechanism to explain ***how*** the particle at the right "knows" what is happening at the left slit. As we will see, the second "fourier transform" approach even provides us with the parameters relevant for this question (namely the distance between the slits), while the first "moments" approach remains silent.

Before proceeding in the next section to the *physics* of interference for single particles, we briefly mention two additional mathematical advantages concerning the second "fourier transform" approach.

First, all the moments $\langle p^n\rangle$ are averages of unbounded quantities, while $\langle\exp\{\frac{i}{\hbar}\hat{p}D\}\rangle$ are averages of bounded quantities. There are problems with unbounded quantities (as pointed out by Mir et al). Infinitesimal changes in $\rho(p)$ can cause very large changes in the moments $\langle p^n\rangle$. To see this, consider a negligible change, $\delta\rho(p)$, in $\rho(p)$. By negligible, we mean there is only a small change in the probability distribution. If we calculate $\delta\langle p^n\rangle = \int \delta\rho(p)p^n dp$, we could get a finite change if $\delta\rho(p)$ differs from zero at a sufficiently large p. In the limit, we could in fact consider $p \to \infty$ and $\delta\rho(p) \to 0$, in such a fashion that Δp^n is finite. Then clearly $\delta\langle p^{n+1}\rangle$ diverges as do all higher moments. The second "fourier transform" approach never has these kinds of problems and is always finite.

The other significant "mathematical" difference concerns the utility of conservation laws. As mentioned in §II.a, while conservation of momenta is certainly maintained for the averages of moments, there is no definite connection between an individual momentum before and after an exchange in this general kind of setup. As we shall see below, the second "fourier transform" approach uncovers an exchange of a new conserved quantity. The conservation law for these quantities can be expressed in a "product-form" rather than a sum (as occurs for ordinary momentum). This product-form conservation law is more relevant for many situations such as a change in relative phase.