Equilibrium wage rigidity in directed search[†]

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Abstract

Matching frictions and downward wage rigidity emerge as equilibrium phenomena in a two-sided labor market where firms sustain variable wage adjustment costs. Firms post wages to attract workers and matches are endogenous. Reducing the wage relative to the wage previously posted is costly to the firm, where the cost is proportional to the size of the proposed cut. Shocks to the firm's profitability may yield an equilibrium wage above what the firm would offer absent proportional adjustment costs. Wage cuts can be partial or full, immediate or delayed, and are non-linear in the shock size. Importantly, wages are sticky even if firms have negligible costs for cutting wages.

Keywords: frictions; matching; sticky wages.

JEL: C70, D40, E30, J30

1 Introduction

A large body of empirical evidence suggests that real wages are not very flexible downward. Firms rarely push through wage cuts (e.g., Agell and Lundborg, 2003; Bewley, 1995; Holzer and Montgomery, 1993), in contrast with the equilibrium prediction of the typical directed search model of frictional labor markets. This kind

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of wage rigidity has also been observed in laboratory economies where workers can direct their search to a firm of their choice (Fehr and Falk, 1999). We ask: can this phenomenon emerge as an equilibrium of frictional labor markets where search can be directed?

Here, we show that equilibrium downward wage rigidity naturally emerges in two-sided labor markets where search is costless and labor market frictions emerge endogenously as in Peters (1984) and Burdett et al. (2001) among many others. To study wage dynamics in a frictional labor market we embed the directed search model, which is static, in an infinitely-lived economy (e.g., as in Albrecht et al., 2006). The model assumes that all matches break at the end of a period, so the existence of equilibrium price rigidities does not rely on the existence of insurance motives when market participants are risk averse and can commit to multi-period contracts (e.g., as in Rudanko, 2009). Moreover, in symmetric equilibrium the wage is unique so the mechanism responsible for price rigidities is not based on assuming a specific focal point when a continuum of bargained wages exists (e.g., as in Hall, 2005).

The model has a two-sided structure as in Peters (1984), where search is costless and unrestricted. In each period many profit-maximizing firms compete in posted wages to attract many utility-maximizing workers who independently direct their search. Lack of coordination among workers is a constant source of frictions, which gives rise to a trade-off for workers and firms: applying to a higher-wage job lowers the probability of being hired, while posting a lower wage reduces the probability of filling a vacancy. This basic market structure forms the basis of the directed search literature of labor markets (e.g., as in Burdett et al., 2001; Shimer, 2005). Its key characteristic is that, in symmetric equilibrium, the distributions of workers across firms *and* of wages reflect the market structure. In particular, labor market frictions emerge endogenously because workers *choose* to direct their search at random based on the distribution of wages that are posted in the period.

The innovation, relative to the directed search models, is the introduction of a variable asymmetric wage adjustment cost. It is assumed that if the firm offers a wage below what it offered in the past, then this wage cut is costly to implement because it generates a variable cost to the firm, which is proportional to the size of the proposed cut. In particular, the cost vanishes as the size of the wage cut shrinks to zero. The central question is: how do firms react when market conditions suddenly change? We identify five levels of the wage posted in the previous period by a firm, which allow full characterization of current wage choices in Markov-perfect equilibrium: an *exit* wage, below which the firm can never attract workers; a *target* wage, which maximizes the firm's expected profit in the absence of any adjustment cost; a *trigger* wage, below which the firm optimally *chooses* not to adjust the current wage; a *break-even* wage, above which there is no wage rigidity; and an *inactive threshold* wage above which the firm prefers to remain temporarily idle; see Figure 1.

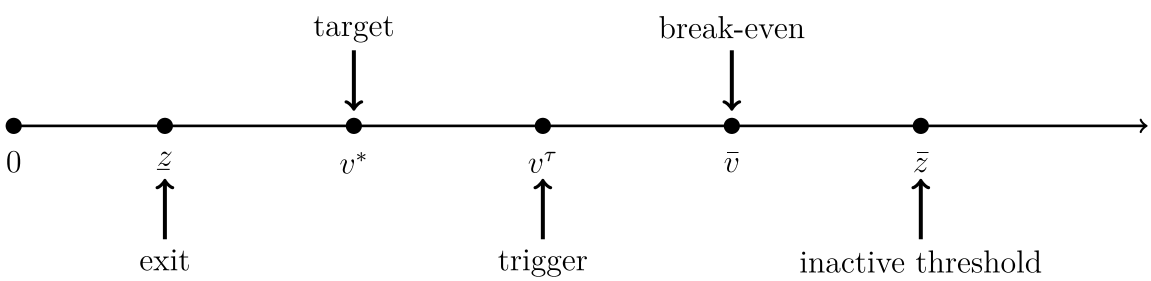

Figure 1: Five key wages

The analysis, reveals that downward wage rigidity generally emerges endogenously, in Nash equilibrium. The resulting wage may lie above temporarily or permanently the 'ideal' target wage. The extent of rigidity nonlinearly depends on the severity of the shock. Firms always react to favorable shocks i.e., shocks that raise their target wage by fully raising the wage to the new target in order to best compete for workers. By contrast, firms will cut wages in reaction to an unfavorable shock only when their target wage sufficiently drops.

Equilibrium wage cuts can be partial or full, immediate or delayed, depending on the shock size. There is a full wage cut when, following the shock, the firm posts exactly the target wage. The wage cut is partial, when the firm instead posts a wage in between the target wage and the previous period's wage. A permanent negative shock induces the firm to optimally cut the wage only if the shock is sufficiently large and even so, full wage cuts are implemented only in extreme cases. This gives rise to three interesting results. First, the decision to cut wages is *negatively* associated with the size of the induced cost. Importantly, wages are sticky even if cutting the wage entails a negligible adjustment cost for the firm. By contrast, wages are cut in response to shocks that significantly lower the target wage, even if this implies high adjustment costs. Second, the size of a wage cut responds non-linearly to the adjustment cost. Generally, firms will push through only partial wage cuts, and only shocks that significantly reduce the firm's profitability will trigger a full wage cut. Third, the firm may optimally choose to temporarily leave the market, and then re-enter it by offering a lower wage. Whether or not this will occur depends on the size of the shock.

Our model assumes that wage adjustment costs arise whenever the firm hires a worker at a wage below the previous period's wage. The adjustment cost is proportional to the size of the wage cut, and is asymmetric, there is no benefit if the wage increases. This modeling choice is economically meaningful: it allows us to capture the important empirical observation that worker's productivity depends on worker's morale, and downward wage rigidity results from employers' desire to avoid harming worker's morale and performance through a wage cut. These observations emerge from multiple surveys (e.g., Bewley, 1995; Campbell and Kamlani, 1997; Kaufman, 1984) as well as a recent field experiment (Kube et al., 2013). Simply put, there is evidence that work effort of existing and newly hired workers alike depends upon workers' morale, and workers' morale is negatively affected by wage cuts, though not necessarily by a low wage level. As a consequence, firms are reluctant to lower wages.

The mechanism explored in this paper supports wage rigidity through a different mechanism a variable cost compared to assuming efficiency wages or fixed adjustment costs which are popular justifications of wage rigidity. In the efficiency wage literature (e.g., Shapiro and Stiglitz, 1984) offering a wage below market disproportionately lowers productivity, so hiring a worker at that lower wage is unprofitable for the firm at any point in time. By contrast, it is the size of the wage cut that matters in our model (not the wage level), and hiring a worker at the lower wage is always profitable for the firm; the wage cut only generates an internal cost.

As in the literature on menu costs, our model assumes an explicit adjustment cost. However, the properties of the adjustment cost in our model are different as compared to other models with explicit adjustment costs. In particular, in the menu costs literature the cost is typically lump-sum and does not vanish as the wage adjustment vanishes (e.g., see Ball and Romer, 1991). This implies that sufficiently small wage cuts are unprofitable. By contrast, in our model the adjustment cost varies proportionately with the size of the cut, and vanishes as the size of the wage cut goes to zero. As a result small wage cuts can in principle be profitable.

The paper proceeds by presenting the model in Section 2. The analysis is laid out in Section 3. Section 4 concludes by offering a discussion of the main features of the model in relation to the literature on the exogenous random matching models and how wage rigidity would be affected if we introduced the possibility of multi-period matches.

2 The model

Time is discrete, $t = 1, 2, \ldots$ There are infinitely-lived agents of two types, $j \in \mathcal{J} = \{1, \ldots, J\}$ firms and $i \in \mathcal{I} = \{1, \ldots, I\}$ homogeneous workers. We assume large markets by letting $J \to \infty$ with $I = rJ$, where $1/r > 0$ denotes market tightness, i.e., the ratio of vacancies to job-seekers. Therefore we work with countably many workers and firms. Every agent discounts the future at geometric rate $\beta \in (0, 1)$. Each period has two stages: wage posting and hiring.[1]

In the first stage of each period t each firm independently posts a wage v_t which is earned by the worker who is hired; there is no cost from posting wages. In the second stage, workers see all posted wages and choose to visit one firm; there is no cost from visiting a firm (see Virag, 2010, for a model where visiting a firm entails a fixed cost).

After matches are realized one vacancy is filled at all firms that have met at least one worker. Matches last only one period, so that separation is certain at the end of the period. Unmatched parties and those matched parties who do not trade obtain zero payoff in the period. If firm j hires a worker on date $t > 1$, then the worker receives v_t and the firm's profit is $\phi_j(v_t; v_{t-1})$, which depends on the current wage v_t and the previous period's wage v_{t-1}. In particular, it is assumed that lowering the wage from the previously posted wage generates a variable cost to the firm. The cost is proportional to the size of the wage cut, so that small wage reductions have a minimal impact on profit of the firm during the period. Moreover, the profit of all future matches remains unaltered. The profit of the firm is formulated by means of the piece-wise profit function

$$\phi_j(v_t; v_{t-1}) := \begin{cases} \tilde{\phi}_j(v_t) - (v_{t-1} - v_t)c_j \mathbf{1}_{v_t < v_{t-1}} & \text{if } t > 1, \\ \tilde{\phi}_j(v_t) & \text{if } t = 1, \end{cases} \tag{1}$$

where

$$\tilde{\phi}_j(v) := a_j - b_j v \geq \phi_j(v; v_{t-1})$$

with $0 < c_j < b_j$. We have $\tilde{\phi}_j(v) \geq 0$ for all $v \in [0, \bar{v}_j]$ where $\bar{v}_j = \dfrac{a_j}{b_j}$ is called the *break-even wage*. Since $\bar{v}_j$ is the crucial element of the analysis, we will normalize

[1]The model and notation reflect the formulation adopted in Kim and Camera (2014), to which we refer the reader for additional details and implied proofs of statements.

$b_j = 1$ for all firms, i.e., we assume perfectly transferable utility. See Figure 2. Since in the initial period $t = 1$ there is no history of previous wages, the profit function is simply $\phi_j(v_1, v_0) = \tilde{\phi}_j(v_1)$.[2]

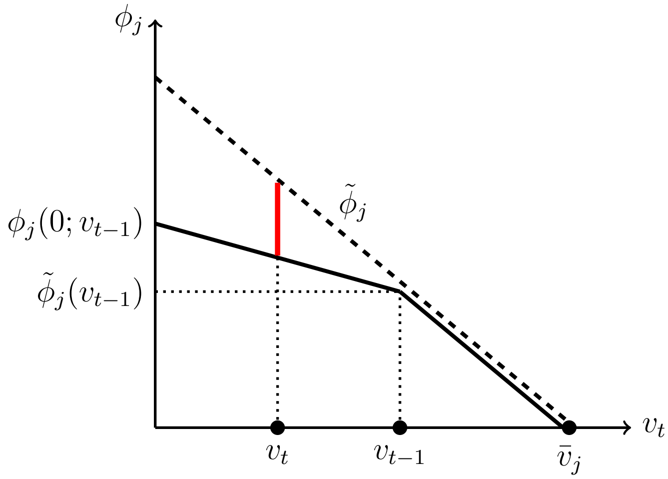

Figure 2: The profit function $\phi_j(v_t; v_{t-1})$

Notes: The red line is the adjustment cost from cutting the wage to v_t from v_{t-1}.

We now discuss the matching function on a generic date t, omitting the argument t when understood. Consider symmetric equilibria, which are the focus in the literature (see Norman, 2015, for a way to justify this focus). Let λ_j denote the expected number of workers who visit firm j in a symmetric outcome where workers are indifferent across all firms and so choose them at random. Given the large size of the market, a standard result is that workers are distributed according to a Poisson with parameter λ_j (Peters, 1984). Consequently we let

$$\mathcal{M}(\lambda_j) := 1 - e^{-\lambda_j}$$

[2]To reduce the notational burden, we will not make this distinction explicit in the remainder of the paper, when no confusion arises. Conceptually, assuming $\phi_j(v_1, v_0) = \tilde{\phi}_j(v_1)$ is equivalent to assuming ϕ_j as in definition (1) for all $t \geq 1$ and $v_0 = 0$.

denote the unconditional probability that firm j trades on date t.[3]

It is assumed that firms who are visited by more than one worker randomly hire one of the workers who are present. Again, the standard result is that the function

$$\mathcal{H}(\lambda_j) := \frac{1 - e^{-\lambda_j}}{\lambda_j} = \frac{\mathcal{M}(\lambda_j)}{\lambda_j},$$

can be used denote the conditional probability that a worker is hired when she visits a firm who expects λ_j workers on date t.

In a symmetric outcome workers must be indifferent across all firms $j \in \mathcal{J}$. Hence, letting v_j denote the wage offered by firm j on date t, the following indifference condition must hold:

$$\mathcal{H}(\lambda_j)v_j = \mathcal{H}(\lambda_k)v_k = U \qquad \text{for all } j, k \in \mathcal{J}.$$

Above, we have used U to denote the expected utility that workers receive on the market in a symmetric outcome. This notation will be convenient later because, given U, we immediately see that the indifference constraint implies that the queue λ_j depends only on v_j. When understood, we will omit the argument v_j from $\lambda_j(v_j)$.

Discussion and interpretation. Our model of adjustment costs can be interpreted as one in which a higher past wage acts like a reduction in productivity. To drive this point home, notice that for the case $v_t < v_{t-1}$, we can reformulate the profit function in our model as follows:[4]

$$\phi(v_t; v_{t-1}) := \bar{v} - cv_{t-1} - (1-c)v_t \propto y_t - v_t$$

[3] One may be tempted to conclude that our analysis can be simplified, at virtually no cost, by imposing an urn-ball matching process. This conclusion is incorrect. The possibility to direct search dramatically changes incentives not only in equilibrium, but also off-equilibrium. See the discussion in Section 4 about the importance of a micro-foundation for the matching function $\mathcal{M}$.

[4] We thank an anonymous referee for suggesting this interpretation and reformulation.

where $y_t := \dfrac{\bar{v} - cv_{t-1}}{1-c}$ is the firm's productivity, which declines in v_{t-1}.

A possible mechanism generating this negative association between past wages and current productivity is tied to the well-received empirical observation that worker's productivity depends on worker's morale. In this case, downward wage rigidity may results from employers' fear that a wage cut would harm worker's morale and productivity. There are several studies providing empirical support to this productivity channel.

The study in Bewley (1995) reports findings from 334 interviews of business owners, managers and human-resource executives from a variety of U.S. small and large companies, during the years 1992-1994. The central conclusion is that the main causes of downward wage rigidity have to do with employers' belief that worker's productivity depends on worker's morale, and wage cuts have a negative effect on morale of existing employees, and also of new hires. Results from the more recent survey in Campbell and Kamlani (1997) confirm this conclusion. Respondents indicated that among the greatest deterrents to wage cuts we find the fear that a wage cut would lead to less effort because it would lower workers' morale. The survey also revealed that workers respond more strongly to a wage cut than to low wages. The study in Kaufman (1984) strikes a similar note for U.K firms, with the authors noting that employers felt that wage reductions, especially those unaccompanied by credible information concerning a financial crisis, would lower morale and effort. In a controlled field experiment that minimized reputation effects and ruled out the possibility of repeated employment, Kube et al. (2013) finds that wage cuts significantly lower workers' productivity, whereas wage increases do not affect it.

It should be clear that the particular combination of assumptions adopted in our model does not allow it to capture all features of the economies analyzed in the aforementioned studies   the presence of incumbent workers, or possible behavioral effects on application probabilities, for example. Our model is simply designed to capture the key operating principle that is common to all the aforementioned studies:

employers' wage choices are influenced by the prospect that a wage cut has adverse effects on workers' productivity. The adjustment cost in our model can be therefore interpreted as workers' productivity decline stemming from a decline in morale. This cost reflects the size of the wage cut but not necessarily the level of the wage, as seen above. The asymmetry in adjustment cost captures the evidence that workers respond asymmetrically to a wage cut as opposed to a wage increase.

3 Equilibrium wages

Here we study the equilibrium behavior of a generic firm, so we omit the indicator j whenever it is understood. We let $v^* \in (0, \bar{v})$ denote the *target wage*, i.e., the wage that the firm optimally posts in the absence of any cost from cutting wages. Since the environment is stationary, v^* corresponds to the static Nash equilibrium wage.

We focus on Markov perfect equilibria, where the firm's posted wage does not depend on the history of play and is not a function of time; the wage depends on the "state" of the profit function and the current market conditions and, in particular, a worker's market utility U, which the firm takes as given. In a stationary environment, wages in Markov perfect equilibrium correspond to wages in static Nash equilibrium (see the discussion in Camera and Kim, 2016).

The analysis identifies five key wage thresholds, which are illustrated in Figure 1: the exit wage $\underline{z}$; the target wage v^*; the trigger wage v^τ; the break-even wage $\bar{v}$; and the inactive threshold wage $\bar{z}$. These thresholds allow us to fully characterize the optimal posted wage, as reported in the following Theorem.

Theorem 1. *Consider Markov-perfect strategies and any period $t > 1$. For any given v_{t-1}, there exists a unique equilibrium sequence of profit-maximizing wages $(v_t^*, v_{t+1}^*, \ldots)$, which satisfies the following properties:*

1. *If $v_{t-1} \leq v^*$, then $v_{t+j}^* = v^*$ for all $j \geq 0$;*

2. *If $v^* < v_{t-1} \leq \bar{v}$, then there exist a "trigger wage" $v^\tau \in (v^*, \bar{v})$ such that*

$$v^*_{t+j} \begin{cases} = v_{t-1} & \text{if } v_{t-1} \in (v^*, v^\tau] \\ \in [v^*, v^\tau) & \text{if } v_{t-1} \in (v^\tau, \bar{v}] \end{cases}$$

for all $j \geq 0$;

3. *If $\bar{v} < v_{t-1}$, then there exists an "inactive threshold" wage $\omega > \bar{z}$ such that*
   - *$v^*_t < v^*$ if $v_{t-1} \in (\bar{v}, \omega)$, and otherwise the firm remains idle in t;*
   - *$v^*_{t+j} = v^*$ for all $j \geq 1$.*

Theorem 1 establishes three facts. First, if the wage last period was below the target wage, then wages immediately and fully adjust to the target v^*, as the firm suffers no adjustment costs. This implies that, without external shocks, optimal wages are constant in Markov equilibrium. They correspond to the target wage v^*, which is simply the market wage in static Nash equilibrium.

The next two points demonstrate the existence of equilibrium wage rigidity, when the firm must sustain an adjustment cost to lower its wage today. Whether or not wages are cut, and by how much, depends on how bigger the previous period's wage v_{t-1} was as compared to the target wage v^*. The second item in Theorem 1 shows that if the wage last period was above the target but below the trigger wage v^τ, then this period's wage will be completely rigid. If instead, last period's wage is between the trigger wage and the break-even wage $\bar{v}$, then the firm will partially lower the current wage, moving it closer to the target v^*. That is, if the wage differential $v_{t-1} - v^*$ is sufficiently small, then the firm will not adjust wages at all at any point in time and will keep the wage at v_{t-1}; otherwise, it will partially adjust the wage, lowering it toward the target.

The third item in Theorem 1 shows that if last period's wage was so large to overtake the break-even wage, then the wage will adjust to the target, but will do so with a delay. The delay is due to the large adjustment cost that must be suffered by the firm today. If v_{t-1} was sufficiently high, then the firm simply stays out of the market today, and re-enters it tomorrow; this is equivalent to posting a zero wage today. Otherwise, there will be "overshooting" of the target wage, because the firm will post a wage below v^* today, only to raise it back up to v^* from tomorrow on. Overshooting stems from the large adjustment cost, which forces the firm to cut current labor costs by offering a low wage today in order to remain profitable.

The remainder of this section is devoted to prove this Theorem through a sequence of Lemmas. We start by establishing a preliminary result about symmetric equilibrium in the static game, which is central to proving existence and uniqueness of symmetric Markov equilibrium in the dynamic game. We then proceed by examining optimal choices in the dynamic game. Finally, we will extend the analysis to the case of unanticipated shocks to firms.

3.1 Preliminaries: equilibrium in the static game

Consider a static game. The firm's profit function depends solely on the wage it chooses to post. For the generic firm j, we have $\phi(v) = \tilde{\phi}(v)$, with ϕ decreasing in v. Consequently, in this section we omit the symbol $\sim$ from $\tilde{\phi}$ for convenience, and simply work with the notation ϕ. Note that, given the normalization $b = 1$, we have $\bar{v} = a$ and $\phi(v) = \bar{v} - v$. For any given market utility U, the firm's maximization problem is thus

$$\max_{v} \ \mathcal{M}(\lambda)(\bar{v} - v) \qquad \text{such that} \qquad \mathcal{H}(\lambda)v = U.$$

From the constraint we obtain $\lambda(v)$ as a function of v, which we can substitute into the objective function to get

$$v^* := \arg\max_v \mathcal{M}(\lambda(v))(\bar{v} - v). \tag{2}$$

Lemma 1 (Wages in Static Nash Equilibrium)**.** *If $\phi(v) = \bar{v} - v$, with $\bar{v} > 0$, then the optimal posted wage v is $v^* = v^*(\bar{v})$. If $\phi_j(v) = \bar{v}_j - v$ for $j = 1, 2$ with $\bar{v}_1 \geq \bar{v}_2$, then $v_1^* \geq v_2^*$.*

There are two important lessons. First, the firm's optimal posted wage v^* is uniquely determined by the break-even wage $\bar{v}$. Second, optimal wages monotonically increase in the break-even wage.

To prove Lemma 1, we study the first order condition of (2). In an interior solution, the optimal value v^* must satisfy the first order condition $G(\bar{v}, v^*) = 0$ where

$$G(\bar{v}, v^*) := \frac{\partial \mathcal{M}(\lambda(v^*))}{\partial \lambda} \times \frac{\partial \lambda(v^*)}{\partial v}(\bar{v} - v^*) - \mathcal{M}(\lambda(v^*)).$$

Note that v^* only depends on $\bar{v}$. Consider the two partial derivatives of $G(\bar{v}, v^*)$, i.e.,

$$\begin{aligned}
&G_{\bar{v}}(\bar{v}, v^*) := \tfrac{\partial \mathcal{M}(\lambda(v^*))}{\partial \lambda} \tfrac{\partial \lambda(v^*)}{\partial v} > 0, \\
&G_{v^*}(\bar{v}, v^*) := \left\{ \tfrac{\partial^2 \mathcal{M}(\lambda(v^*))}{\partial \lambda^2} \left(\tfrac{\partial \lambda(v^*)}{\partial v} \right)^2 + \tfrac{\partial \mathcal{M}(\lambda(v^*))}{\partial \lambda} \tfrac{\partial^2 \lambda(v^*)}{\partial v^2} \right\} (\bar{v} - v^*) - 2 \tfrac{\partial \mathcal{M}(\lambda(v^*))}{\partial \lambda} \tfrac{\partial \lambda(v^*)}{\partial v} < 0.
\end{aligned}$$

The implicit function theorem, the characterization of the function $\lambda(v)$ (see the Appendix), and the properties of $\mathcal{M}$ imply

$$\frac{\partial v^*}{\partial \bar{v}} = -\frac{G_{\bar{v}}(\bar{v}, v^*)}{G_{v^*}(\bar{v}, v^*)} > 0.$$

Now note that $G_{v^*}(\bar{v}, v^*)$, which is negative, represents the second order condition for the problem in (2). Therefore, v^* uniquely maximizes the firm's profit.

Lemma 1 is illustrated in Figure 3; it displays the optimal posted wage v_j^* for two firms $j = 1, 2$, which depends only on the break-even wage $\bar{v}_j$ that firm j can offer. The left panel shows why the normalization $b_j = 1$ is without loss in generality; there, we have $a_2 < a_1$ and $b_2 < b_1 = 1$ such that $\frac{a_2}{b_2} = \frac{a_1}{b_1} = \bar{v}_1 = \bar{v}_2$, so the two optimal posted wages coincide. The right panel shows two firms with normalized $b_2 = b_1 = 1$ and $a_2 > a_1$ so $\bar{v}_2 > \bar{v}_1$.

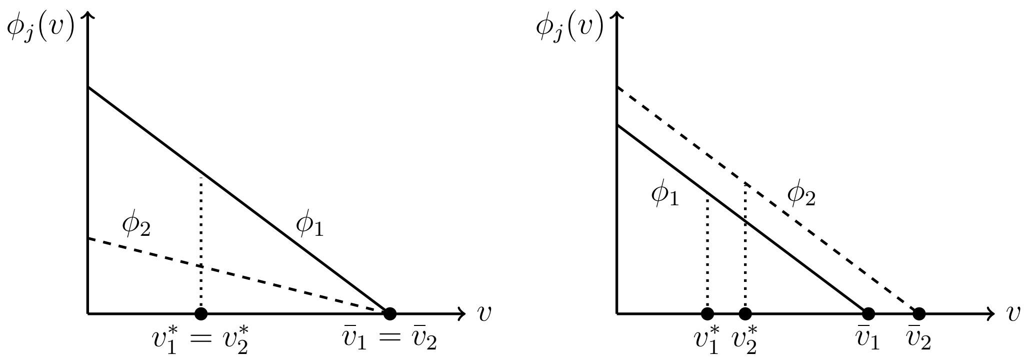

Figure 3: The optimal posted wage v_j^*

3.2 The target wage in the dynamic game

Consider the dynamic game. We derive the sequence of target wages and show it is constant. Recall that when workers are indifferent across firms, then demand on date t is determined by $\lambda(v_t)$. The **expected profit** function on date t is

$$\mathcal{M}(v_t)\phi(v_t; v_{t-1}) = \mathcal{M}(v_t)\tilde{\phi}(v_t) - \mathcal{M}(v_t)(v_{t-1} - v_t)c\mathbf{1}_{v_t < v_{t-1}},$$

where for convenience we have omitted the argument λ from the function $\mathcal{M}(\lambda(v_t))$.

The central difference from the static game is that the expected profit function $\mathcal{M}(v_t)\phi(v_t; v_{t-1})$ is not differentiable at $v_t = v_{t-1}$ because ϕ has a kink at that point.

The kink is the consequence of the temporary decline in profit due to the variable adjustment cost. Hence, define the partial derivative

$$F(v_t; v_{t-1}) := \frac{\partial\{\mathcal{M}(v_t)\tilde{\phi}(v_t) - \mathcal{M}(v_t)(v_{t-1} - v_t)c\}}{\partial v_t},$$

where we note that $F(v_t; v_{t-1}) = \dfrac{\partial\{\mathcal{M}(v_t)\phi(v_t; v_{t-1})\}}{\partial v_t}$ for $v_t < v_{t-1}$.

Let $\Pi(v_{t-1})$ denote the firm's maximal lifetime utility, or maximal **payoff**, on date t given v_{t-1}. This is the maximal discounted sum of expected profits. Since the environment is stationary we use the recursive formulation

$$\begin{aligned}\Pi(v_{t-1}) &= \max_{v_t}\{\mathcal{M}(v_t)\phi(v_t; v_{t-1}) + \beta\Pi(v_t)\} \\ &= \max_{v_t} \Pi(v_t; v_{t-1}),\end{aligned} \tag{3}$$

where

$$\Pi(v_t; v_{t-1}) := \mathcal{M}(v_t)\phi(v_t; v_{t-1}) + \beta\Pi(v_t).$$

Let $v_t^* := v_t^*(v_{t-1})$ be a solution to $\max\limits_{v_t} \Pi(v_t; v_{t-1})$. In what follows we will show that the maximizer v_t^* is unique and is in the interior of $[0, \bar{v}]$.

Lemma 2 (Target wages)**.** *There exists a unique solution* $v_t = v^* \in (0, \bar{v})$*, which solves the firm's problem* $\max\limits_{v_t} \Pi(v_t; v_{t-1})$ *in each* $t = 1, 2, \ldots$

Proof of Lemma 2. Recall that in $t = 1$ we have $\phi(v_1; v_0) = \tilde{\phi}(v_1)$, following the definition in (1). Since the environment is stationary, consider an outcome such that $v_t^* = v^* \in [0, \bar{v}]$ for all t. We have

$$\Pi(v^*) = \left.\frac{\mathcal{M}(v_t)\phi(v_t; v_{t-1})}{1-\beta}\right|_{v_t = v_{t-1} = v^*} = \frac{\mathcal{M}(v^*)\tilde{\phi}(v^*)}{1-\beta}.$$

From earlier results we know that, for all v such that $\mathcal{M}(v) > 0$, the function

$\mathcal{M}(v)\tilde{\phi}(v)$ is strictly concave and differentiable. That is, when the firm is in the market, then his expected profit function is smooth and strictly concave (Peters, 1984). Consequently, there exists a unique interior optimum v^* that satisfies

$$\frac{\partial\{\mathcal{M}(v^*)\tilde{\phi}(v^*)\}}{\partial v} = 0.$$

In fact, it is easy to see that the stationary sequence $\{v^*\}$ is the unique equilibrium. Consider any sequence where at least one wage is off-target, i.e., $\{v_t\}_{t=1}^{\infty} \neq \{v^*\}$. From above, we have that v^* is the unique solution to $\max_v \mathcal{M}(v)\tilde{\phi}(v)$. To show that non-stationary sequences $\{v_t\}_{t=1}^{\infty}$ are also suboptimal, using the definition of ϕ in equation (1) we have

$$\begin{aligned}\Pi(v^*) = \frac{\mathcal{M}(v^*)\tilde{\phi}(v^*)}{1-\beta} &= \sum_{t=1}^{\infty} \beta^{t-1}\mathcal{M}(v^*)\tilde{\phi}(v^*) \\ &> \sum_{t=1}^{\infty} \beta^{t-1}\mathcal{M}(v_t)\tilde{\phi}(v_t) \geq \sum_{t=1}^{\infty} \beta^{t-1}\mathcal{M}(v_t)\phi(v_t; v_{t-1}).\end{aligned}$$

□

Lemma 2 proves that in Markov equilibrium wages coincide with the target wage v^* in each period. The open question is: what wage would the firm optimally post currently, if it last period it posted a wage that does not correspond to the current target? Hence, in what follows, we discuss the optimal wage v_t^* for any off-target case $v_{t-1} \neq v^*$. For convenience, we define

$$\tilde{F}(v) := \frac{\partial\{\mathcal{M}(v)\tilde{\phi}(v)\}}{\partial v},$$

so that at the optimum $v = v^*$ we have $\tilde{F}(v^*) = 0$.

Given the formulation for ϕ in equation (1) we need to consider left and right derivatives, which we respectively denote

$$f'_-(x_0) := \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0}, \qquad \text{and} \qquad f'_+(x_0) := \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0}.$$

As usual, if $f'_-(x_0) = f'_+(x_0) < \infty$, then f is differentiable at x_0, in which case we define $f'(x_0) := f'_-(x_0) = f'_+(x_0)$.

3.3 The optimal wage when the previous wage is off-target

In this section we show how to calculate the optimum wage v_t^* when the previous wage was off-target. There are several cases to consider, depending on whether v_{t-1} is (i) above or below the target wage v^* and (ii) near to or far from v^*.

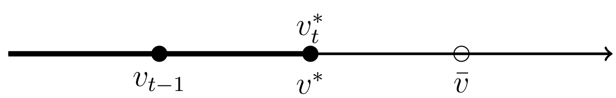

Figure 4: Case 1 The optimal wage v_t^* when $v_{t-1} < v^*$.

Lemma 3 (Full adjustment)**.** *If $v_{t-1} < v^*$ on date t, then $v_t^* = v^*$ uniquely solves $\max_{v_t} \Pi(v_t; v_{t-1})$, hence $v_{t+j}^* = v^*$ for all $j \geq 1$.*

Proof of Lemma 3. Let $v_{t-1} < v^*$ on date t. By definition $\tilde{\phi}(v^*) = \phi(v^*; v_{t-1})$. The firm's optimal choice is $v_t^* = v^*$ because the maximum payoff is achieved at v^*: $\Pi(v^*) = \mathcal{M}(v^*)\tilde{\phi}(v^*) + \beta\Pi(v^*) = \mathcal{M}(v^*)\phi(v^*; v_{t-1}) + \beta\Pi(v^*)$. $\square$

Lemma 3 establishes that if the firm yesterday posted a wage that is below the current target, then it is optimal to raise the wage back to the target level. This is so because raising the wage increases the firm's expected profit and there is no cost. Hence, calculation of equilibrium wages follows the standard analysis.

Things are different when the firm has previously posted a wage *above* target. Here the firm faces a profit decline in period t due to the cost. To discuss this case, we define the left derivative

$$\Pi'_{-}(v; v_{t-1}) := \lim_{v_t \to v-} \frac{\Pi(v_t; v_{t-1}) - \Pi(v; v_{t-1})}{v_t - v}$$

and similarly for the opposite right derivative $\Pi'_{+}(v; v_{t-1})$.

The first step to understand optimal wage setting when $v_{t-1} > v^*$, is to establish where v_t^* lies relative to the state v_{t-1}.

Lemma 4 (Direction of adjustment)**.** *If $v_{t-1} > v^*$ on date t, then the firm optimally chooses $v_t^* \leq v_{t-1}$.*

Proof of Lemma 4. The proof is by contradiction. Suppose that there exist *some* $v_{t-1} > v^*$ such that $v_t^* > v_{t-1}$ is optimal. In fact, we have

$$\begin{aligned}
\Pi(v_{t-1}; v_{t-1}) &= \mathcal{M}(v_{t-1})\phi(v_{t-1}; v_{t-1}) + \beta\Pi(v_{t-1}) \\
&\geq \mathcal{M}(v_{t-1})\phi(v_{t-1}; v_{t-1}) + \beta\Pi(v_t^*) = \mathcal{M}(v_{t-1})\tilde{\phi}(v_{t-1}) + \beta\Pi(v_t^*) \\
&> \mathcal{M}(v_t^*)\tilde{\phi}(v_t^*) + \beta\Pi(v_t^*) = \mathcal{M}(v_t^*)\phi(v_t^*; v_{t-1}) + \beta\Pi(v_t^*) \\
&= \Pi(v_t^*; v_{t-1}),
\end{aligned}$$

which contradicts that v_t^* is optimal. The first inequality follows from the fact that the maximal payoff $\Pi(v_{t-1})$ can be no less than $\Pi(v_t^*)$ since by conjecture $v_t^* > v_{t-1}$. The second inequality follows from the concavity of $\mathcal{M}(v_t)\tilde{\phi}(v_t)$ which is maximized at $v_t = v^*$ and the observation that $\phi(v_t^*; v_{t-1}) = \tilde{\phi}(v_t^*)$ when $v_t^* > v_{t-1}$. $\square$

A firm that has (incorrectly) posted wages above the target v^* has an incentive to revert back to the optimal target and not to compound the problem by moving further away from it. Hence, the firm will not offer more than v_{t-1} on date t. The central question is whether the firm will find it optimal to *fully or partially* adjust wages back to the target v^*, or not adjust them at all.

To proceed, define the *trigger wage* $v^\tau > v^*$ as follows. If the firm yesterday posted $v_{t-1} = v^\tau$, then the firm is indifferent to marginally cut its wage below v_{t-1}. Hence, v^τ is the solution to

$$\frac{\tilde{F}(v)}{1-\beta} + \mathcal{M}(v)c = 0.$$

The term $\dfrac{\tilde{F}(v)}{1-\beta}$ accounts for the marginal revenue decline from not altering wages, while $\mathcal{M}(v)c$ accounts for the marginal savings from avoiding temporary profit losses.

Lemma 5. *A unique solution v^τ to the equation $\dfrac{\tilde{F}(v)}{1-\beta} + \mathcal{M}(v)c = 0$ exists in $(v^*, \bar{v})$.*

Proof of Lemma 5. If we evaluate it at $v = v^*$, then

$$\frac{\tilde{F}(v^*)}{1-\beta} + \mathcal{M}(v^*)c = \mathcal{M}(v^*)c > 0,$$

since $\tilde{F}(v^*) = 0$ by definition. If we evaluate it at the break-even wage $v = \bar{v}$, then

$$\begin{aligned}\frac{\tilde{F}(\bar{v})}{1-\beta} + \mathcal{M}(\bar{v})c &= \frac{\mathcal{M}'(\bar{v})\tilde{\phi}(\bar{v})}{1-\beta} - \frac{\mathcal{M}(\bar{v})}{1-\beta} + \mathcal{M}(\bar{v})c \\ &= \left(c - \frac{1}{1-\beta}\right)\mathcal{M}(\bar{v}) < 0,\end{aligned}$$

where we have used the fact that $\tilde{\phi}(\bar{v}) = 0$ and $1 > c$ (by definition). By the continuity of the evaluated function, we can find a solution $v^\tau \in (v^*, \bar{v})$. Uniqueness of the solution v^τ follows from noticing that $\mathcal{M}(v)$ is strictly concave and monotonically increasing in $v \in [v^*, \bar{v}]$. That is,

$$\tilde{F}'(v) + \mathcal{M}'(v)(1-\beta)c = \mathcal{M}''(v)\tilde{\phi}(v) - \mathcal{M}'(v)[2 - (1-\beta)c] < 0.$$

Hence $\dfrac{\tilde{F}(v)}{1-\beta} + \mathcal{M}(v)c$ is monotonically decreasing, and the solution is unique. $\square$

Therefore $\dfrac{\tilde{F}(v)}{1-\beta} + \mathcal{M}(v)c \geq 0$ for all $v \in (v^*, v^\tau]$.

Lemma 6 (Wage rigidity)**.** *If $v_{t-1} \in (v^*, v^\tau]$ on date t, then $v_t^* = v_{t-1}$ uniquely solves the firm's problem $\max_{v_t} \Pi(v_t; v_{t-1})$. Hence, $v_{t+j}^* = v_{t-1}$ for all $j \geq 1$.*

Proof of Lemma 6. Let $v_{t-1} \in (v^*, v^\tau]$ on date t. There are three possible cases: $v_{t+1}^* > v_t^*$, $v_{t+1}^* < v_t^*$, and $v_{t+1}^* = v_t^*$. We will show that the first two cases are impossible, while the third case represents the solution to the firm's problem.
(i) $v_{t+1}^* > v_t^*$: By Lemma 4, we must have $v_t^* < v^*$. Note that

$$\begin{aligned}
\Pi'_-(v^*; v_{t-1}) =& \lim_{v \to v^*-} \frac{\Pi(v; v_{t-1}) - \Pi(v^*; v_{t-1})}{v - v^*} \\
=& \lim_{v \to v^*-} \frac{\mathcal{M}(v)\phi(v; v_{t-1}) - \mathcal{M}(v^*)\phi(v^*; v_{t-1}) + \beta\Pi(v) - \beta\Pi(v^*)}{v - v^*} \\
=& F(v^*; v_{t-1}) > 0.
\end{aligned}$$

In the third line we have used the fact that $\Pi(v) = \Pi(v^*)$ for any $v \leq v^*$; and the inequality comes from Lemma 1. However, it contradicts that $v_t^* < v^*$ because the firm can marginally increase the wage and improve its expected payoff. Hence $v_{t+1}^* > v_t^*$ is inconsistent with optimality.
(ii) $v_{t+1}^* < v_t^*$: By the recursive nature of the problem, we must have $v_t^* < v_{t-1}$. To show that this inequality holds, suppose it is not true. Then, by Lemma 4 we can only have $v_t^* = v_{t-1}$; if so, then the firm would face the same problem tomorrow, so we must have $v_{t+1}^* = v_t^*$. But this is in contradiction with the conjecture $v_{t+1}^* < v_t^*$.

Given that $v_t^* < v_{t-1}$, note that since $\Pi(v_{t-1}) = \max_{v_t}\{\mathcal{M}(v_t)\phi(v_t; v_{t-1}) + \beta\Pi(v_t)\}$ and $v_t^* < v_{t-1}$, then $\Pi'(v_{t-1})$ is well defined and equals to $-\mathcal{M}(v_t^*)c$ by the envelope theorem. Note that

$$
\begin{aligned}
\Pi'_{-}(v_{t-1}; v_{t-1}) =& \lim_{v \to v_{t-1}-} \frac{\Pi(v; v_{t-1}) - \Pi(v_{t-1}; v_{t-1})}{v - v_{t-1}} \\
=& \lim_{v \to v_{t-1}-} \frac{\mathcal{M}(v)\phi(v; v_{t-1}) - \mathcal{M}(v_{t-1})\phi(v_{t-1}; v_{t-1}) + \beta\Pi(v) - \beta\Pi(v_{t-1})}{v - v_{t-1}} \\
=& \tilde{F}(v_{t-1}) + \mathcal{M}(v_{t-1})c - \beta\mathcal{M}(v_t^*)c \\
>& \tilde{F}(v_{t-1}) + \mathcal{M}(v_{t-1})c - \beta\mathcal{M}(v_{t-1})c > 0.
\end{aligned}
$$

In the third line we have used the fact that $\Pi'(v_{t-1}) = -\mathcal{M}(v_t^*)c$; the fourth line comes from $v_t^* < v_{t-1}$; and the last inequality comes from the fact that $\frac{\tilde{F}(v)}{1-\beta} + \mathcal{M}(v)c$ is monotonically decreasing and $v_{t-1} < v^\tau$. Hence we find a contradiction for $v_t^* < v_{t-1}$. Therefore $v_{t+1}^* < v_t^*$ is inconsistent with optimality.
(iii) $v_{t+1}^* = v_t^*$: If $v_{t+1}^* = v_t^*$ is optimal, then $v_{t+j}^* = v_t^*$ should be optimal for all $j \geq 1$ by the recursive nature of the problem. For any $v \in (v^*, v_{t-1})$, the marginal payoff is

$$
\begin{aligned}
\frac{\partial}{\partial v}\left[\mathcal{M}(v)\phi(v; v_{t-1}) + \frac{\beta}{1-\beta}\mathcal{M}(v)\tilde{\phi}(v)\right] =& F(v, v_{t-1}) + \frac{\beta}{1-\beta}\tilde{F}(v) \\
=& \frac{1}{1-\beta}\tilde{F}(v) + \mathcal{M}(v)c - \mathcal{M}'(v)(v_{t-1} - v)c.
\end{aligned}
$$

The limit of the above first order condition as $v \to v_{t-1}$ from the left is

$$
\frac{1}{1-\beta}\tilde{F}(v_{t-1}) + \mathcal{M}(v_{t-1})c > 0.
$$

Together with Lemma 4, $v_t^* = v_{t-1}$ is the unique solution to the firm's problem. $\square$

Lemma 6 establishes that a firm finding itself with wages slightly above target *will not* adjust wages at all, neither presently nor in the future. The firm has no incentive to adjust the current wage because the cost is larger than the benefit. By the recursive nature of the problem, the firm has no incentive to perform adjustments in the future, either. Figure 5 provides an illustration.

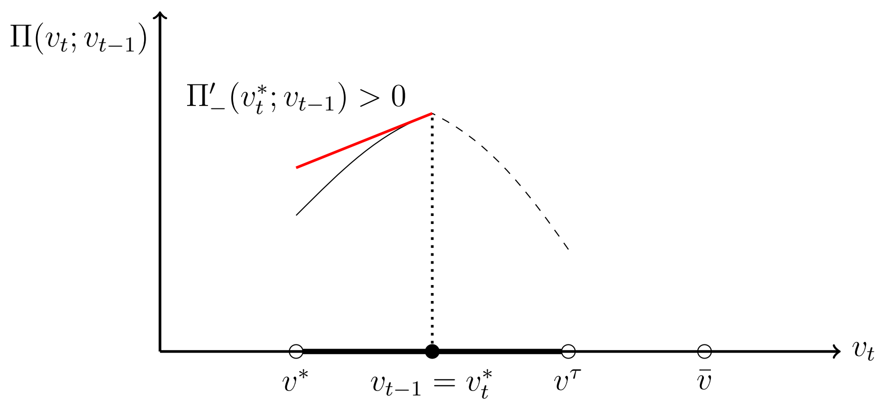

Figure 5: Case 2: The kink at v_t^* when $v_{t-1} \in (v^*, v^\tau)$

Yesterday's wage in the figure is above the optimal target, although not so much, i.e., we have $v_{t-1} \in (v^*, v^\tau)$. The curved thin line represents the expected profit $\Pi(v_t; v_{t-1})$, which reveals that the firm benefits from raising the wage from v^* because the marginal profit is positive, declining as we move towards v_{t-1}. The straight thick line represents the marginal profit when $v_t = v_{t-1}$, which is still positive, i.e., $\Pi'_-(v_t; v_{t-1}) > 0$. So, why does the firm choose to stop at this point, and post $v_t^* = v_{t-1}$, instead of moving further up? Going beyond v_{t-1} causes a *discrete* drop in the firm's marginal profit, because at v_{t-1} the firm suffers a temporary adjustment cost that generates a kink in the payoff. This is illustrated by the dashed, declining curve. Hence, the firm should neither lower the wage back down to the target v^*, nor increase it above v_{t-1}. The firm should simply offer the same wage it offered yesterday, $v_t^* = v_{t-1}$.

It is important to emphasize that the firm does not face exogenous restrictions to its ability to offer wages because the choice set remains $[0, \bar{v}]$. Lemma 6 shows that the firm optimally *chooses* not to lower wages down to the target level v^* because doing so would entail an extra-marginal cost. The Lemma proves that when wages are slightly above target, then this extra-marginal cost is greater than the revenue decline associated with adjusting the wage back to target.

The question that remains to be discussed is: what happens to wages when the firm's wages are significantly above target?

Lemma 7 (Sticky wages)**.** *If* $v_{t-1} \in (v^\tau, \bar{v})$ *on date* t*, then there exists a unique solution to the firm's problem* $\max_{v_t} \Pi(v_t; v_{t-1})$ *such that* $v_t^* \in (v^*, v^\tau)$*. Hence,* $v_{t+j}^* = v_t^*$ *for all* $j \geq 1$*. Furthermore,* $\dfrac{\partial v_t^*}{\partial v_{t-1}} < 0$*.*

Proof of Lemma 7. The left-sided first order condition evaluated at the trigger wage $v_t = v^\tau$ is

$$
\begin{aligned}
\Pi'_-(v^\tau; v_{t-1}) &= \lim_{v \to v^\tau -} \frac{\Pi(v; v_{t-1}) - \Pi(v^\tau; v_{t-1})}{v - v^\tau} \\
&= \lim_{v \to v^\tau -} \frac{\mathcal{M}(v)\phi(v; v_{t-1}) - \mathcal{M}(v^\tau)\phi(v^\tau; v_{t-1}) + \beta\Pi(v) - \beta\Pi(v^\tau)}{v - v^\tau} \\
&= F(v^\tau; v_{t-1}) + \frac{\beta \tilde{F}(v^\tau)}{1-\beta} \\
&= \frac{\tilde{F}(v^\tau)}{1-\beta} + \mathcal{M}(v^\tau)c - \mathcal{M}'(v^\tau)(v_{t-1} - v^\tau)c \\
&= -\mathcal{M}'(v^\tau)(v_{t-1} - v^\tau)c < 0.
\end{aligned}
$$

In the third line we have used the fact that $\Pi(v) = \dfrac{\mathcal{M}(v)\phi(v;v)}{1-\beta}$ for $v \in (v^*, v^\tau]$ by Lemma 6; in the fourth line we have decomposed $F(v^\tau; v_{t-1})$; and in the last line we have used $\tilde{F}(v^\tau) + (1-\beta)\mathcal{M}(v^\tau)c = 0$. Therefore $v_t^* < v^\tau$.

The right-sided first order condition evaluated at $v_t = v^*$ is

$$
\begin{aligned}
\Pi'_+(v^*; v_{t-1}) &= \lim_{v \to v^* +} \frac{\Pi(v; v_{t-1}) - \Pi(v^*; v_{t-1})}{v - v^*} \\
&= \lim_{v \to v^* +} \frac{\mathcal{M}(v)\phi(v; v_{t-1}) - \mathcal{M}(v^*)\phi(v^*; v_{t-1}) + \beta\Pi(v) - \beta\Pi(v^*)}{v - v^*} \\
&= F(v^*; v_{t-1}) + \frac{\beta \tilde{F}(v^*)}{1-\beta} \\
&= \mathcal{M}(v^*)c - \mathcal{M}'(v^*)(v_{t-1} - v^*)c > 0.
\end{aligned}
$$

The third line is derived from $\Pi(v) = \dfrac{\mathcal{M}(v)\phi(v;v)}{1-\beta}$ for $v \in [v^*, v^\tau]$, and the inequality comes from the fact that v^* satisfies

$$\mathcal{M}(v^*) - \mathcal{M}'(v^*)(\bar{v} - v^*) = 0, \tag{4}$$

which is the first order condition for the static game equilibrium. Hence we also have $v_t^* \geq v^*$. Note that the left and right derivatives of $\Pi(v; v_{t-1})$ at $v \in (v^*, v^\tau)$ are the same (because the function is differentiable at that point), so we can find $v_t^* \in (v^*, v^\tau)$ by using the first order condition $\Pi'(v; v_{t-1})$ evaluated at $v = v_t^*$, i.e.,

$$F(v_t^*; v_{t-1}) + \frac{\beta \tilde{F}(v_t^*)}{1-\beta} = \frac{\tilde{F}(v_t^*)}{1-\beta} + \mathcal{M}(v_t^*)c - \mathcal{M}'(v_t^*)(v_{t-1} - v_t^*)c = 0.$$

We have $\dfrac{\partial v_t^*}{\partial v_{t-1}} = \dfrac{\mathcal{M}'(v_t^*)c}{\frac{\tilde{F}'(v_t^*)}{1-\beta} + 2\mathcal{M}'(v_t^*)c - \mathcal{M}''(v_t^*)(v_{t-1} - v_t^*)c} < 0$, since the denominator is negative. The denominator is negative because

$$\begin{aligned}
&\frac{\partial}{\partial v_t^*}\left[\tfrac{\tilde{F}(v_t^*)}{1-\beta} + \mathcal{M}(v_t^*)c - \mathcal{M}'(v_t^*)(v_{t-1} - v_t^*)c\right] \\
&= \tfrac{\tilde{F}'(v_t^*)}{1-\beta} + 2\mathcal{M}'(v_t^*)c - \mathcal{M}''(v_t^*)(v_{t-1} - v_t^*)c < \tilde{F}'(v_t^*) + 2\mathcal{M}'(v_t^*) - \mathcal{M}''(v_t^*)(v_{t-1} - v_t^*) \\
&= \mathcal{M}''(v_t^*)[\tilde{\phi}(v_t^*) - (v_{t-1} - v_t^*)] = \mathcal{M}''(v_t^*)\tilde{\phi}(v_{t-1}) < 0.
\end{aligned}$$

The second line is derived from the fact that $\tilde{F}'(v_t^*) < 0$ and $2\mathcal{M}'(v_t^*) - \mathcal{M}''(v_t^*)(v_{t-1} - v_t^*) > 0$ with $c < 1$.

Due to the recursive nature of the firm's problem and Lemma 6, we have $v_{t+j} = v_t^*$ for all $j = 1, 2, \ldots$ $\square$

Consider the case when $v_{t-1} \in (v^\tau, \bar{v})$, which roughly corresponds to the case when the firm yesterday posted a wage that is considerably above the target wage v^*. Lemma 7 shows that the firm will lower the wage, posting a value v_t^* in between the target wage v^* and the trigger wage v^τ. In this case the temporary loss from adjusting the wage is dominated by the gain in expected revenue. By the recursive nature of the problem, the firm has no incentive to perform further adjustments in the future and will keep wages constant thereafter. Wages are adjusted downward only in the current period because further adjustments would be costly; hence, progressive adjustment is inherently suboptimal for the firm.

Figure 6 provides an illustration. Last period the firm posted a wage v_{t-1} that is above the trigger wage v^τ, but still below the break-even wage $\bar{v}$. To maximize the payoff, the firm lowers the wage toward v^*, stopping at v_t^*.

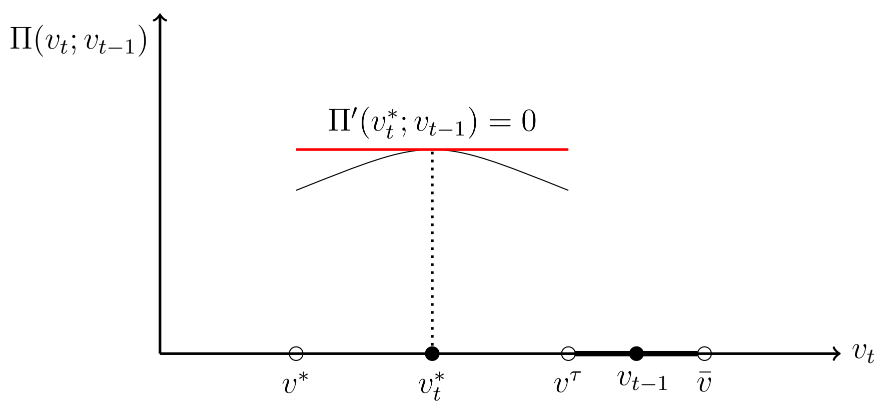

Figure 6: Case 3: No kink at v_t^* when $v_{t-1} \in (v^\tau, \bar{v})$

The equilibrium wage for the case studied in Lemma 7, and depicted in Figure 6, has an interesting property. Larger departures of the previous period's wage v_{t-1} from the target wage v^* result in a larger wage cut, i.e., a smaller current wage v_t^*. In other words, the firm optimally chooses to adjust the wage closer and closer to the target wage v^* as v_{t-1} moves further and further away from the trigger level v^τ.

Why would we observe a current wage offer v_t^* that *decreases* in the firm's previous wage offer v_{t-1}? The reason is that a change in wage has two effects. It alters realized profit conditional on match with a worker because it alters the cost of labor (intensive margin). It also alters expected profits because it alters the expected number of workers λ, hence the probability of match (extensive margin). By cutting the wage relative to previous period's wage, the firm suffers an adjustment cost. As discussed in Section 2, this adjustment cost be interpreted as a productivity decline.

A large wage cut i.e., a wage cut bringing v_t^* close to the target v^* allows the firm to counteract this productivity decline by decreasing its labor cost. All else equal, larger values of v_{t-1} bring about a larger productivity decline, which magnifies the firm's desire do reduce labor costs, inducing it to lower the wage even closer to the target v^*. The firm does not cut the wage all the way to v^* because doing so greatly reduces the matching probability, causing an expected revenue loss that outweighs the savings from lower labor costs.

Now consider $v_{t-1} = \bar{v}$, i.e., the firm posted the break-even wage. Here, there will be full adjustment to the target wage v^* because

$$\begin{aligned} v_t^* &= \arg\max_{v} \mathcal{M}(v)\phi(v;\bar{v}) + \beta\Pi(v) \\ &= \arg\max_{v\leq\bar{v}} (1-c)\mathcal{M}(v)\tilde{\phi}(v) + \beta\Pi(v) \\ &= \arg\max_{v\leq\bar{v}} \mathcal{M}(v)\tilde{\phi}(v) + \beta\Pi(v) = v^*. \end{aligned}$$

The second equality follows from two observations. First, the firm chooses a wage $v \leq \bar{v}$ (by Lemma 4). Second, when $v_{t-1} = \bar{v}$ we have $\phi(v;\bar{v}) = (1-c)\tilde{\phi}(v)$ (definitions of profit functions). This second observation is crucial: it implies that a firm that posted the break-even wage last period, will currently suffer an adjustment cost for *any* profitable wage it will choose to post. The question is then, how large a cost should the firm suffer conditional on hiring. Note that in this case the profit function has no kink but it is linear. If so, then marginal profits are zero at $v_t^* = v^*$, as when there are no adjustment costs at all because the only optimality element that matters when the profit function is linear is the break even wage (Lemma 1). This implies the third equality, i.e., the solution to the maximization problem is equivalent to the unique solution to the unconstrained problem.

Now consider the case when the wage from the previous period, v_{t-1}, is so large that is lies above the break-even wage $\bar{v}$. This corresponds to the third item in

Theorem 1. Let $\underline{z} < v^*$ denote the *exit wage*; any firm that offers $v \leq \underline{z}$ cannot attract workers so the firm exits the market. Recall that the firm takes a worker's market utility U as given. Consequently, workers' indifference implies that $\underline{z} = U$ and if $v > \underline{z}$, then $\mathcal{H}(\lambda)v = U$.

Let $\bar{z} > \bar{v}$ be the *inactive threshold*, i.e., if $v_{t-1} \geq \bar{z}$, then the firm cannot earn a profit by offering something above $\bar{v}$. Moreover, the adjustment cost is so large that the firm cannot be profitable even if it offers a wage below $\bar{v}$. Hence, if $v_{t-1} \geq \bar{z}$, then it is optimal for a firm to be inactive in t. Here, $\bar{z}$ uniquely solves $\phi(\bar{z}; \underline{z}) = 0$.

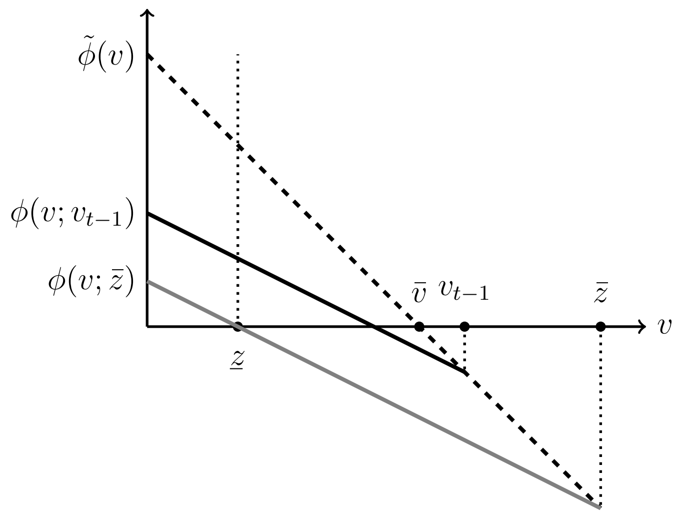

Figure 7: The profit function ϕ in period t, when $v_{t-1} > \bar{v}$

Lemma 8. *Let $v_{t-1} \in (\bar{v}, \bar{z})$ on date $t > 1$. There exists a unique solution to the firm's problem $\max_{v_t} \Pi(v_t; v_{t-1})$ such that $v_t^* < v^*$. Hence, $v_{t+j}^* = v^*$ for all $j \geq 1$.*

Proof. Let $v_{t-1} \in (\bar{v}, \bar{z})$. The left-sided first order condition evaluated at $v_t = v^*$ is

$$
\begin{aligned}
\Pi'_{-}(v^*; v_{t-1}) &= \lim_{v \to v^*-} \frac{\Pi(v; v_{t-1}) - \Pi(v^*; v_{t-1})}{v - v^*} \\
&= \lim_{v \to v^*-} \frac{\mathcal{M}(v)\phi(v; v_{t-1}) - \mathcal{M}(v^*)\phi(v^*; v_{t-1}) + \beta\Pi(v) - \beta\Pi(v^*)}{v - v^*} \\
&= \lim_{v \to v^*-} \frac{\mathcal{M}(v)\phi(v; v_{t-1}) - \mathcal{M}(v^*)\phi(v^*; v_{t-1})}{v - v^*} = F(v^*; v_{t-1}) \\
&= \tilde{F}(v^*) + \mathcal{M}(v^*)c - \mathcal{M}'(v^*)(v_{t-1} - v^*)c < 0.
\end{aligned}
$$

The third line is obtained by noticing that the continuation payoff on date t is $\Pi(v) = \Pi(v^*) = \dfrac{\mathcal{M}(v^*)\phi(v^*; v^*)}{1-\beta}$ for all $v < v^*$ by Lemma 3; in the last line, we have decomposed $F(v^*; v_{t-1})$; and in the inequality is obtained by $\tilde{F}(v^*) = 0$, and $v_{t-1} > \bar{v}$ with the equation (4). Therefore, it is optimal for the firm to post $v < v^*$. As the firm cannot attract workers if $v_t \leq \underline{z}$, we have

$$\Pi(v; v_{t-1}) = \mathcal{M}(v)\phi(v; v_{t-1}) + \beta\Pi(v) > \mathcal{M}(x)\phi(x; v_{t-1}) + \beta\Pi(x) = \Pi(x; v_{t-1})$$

for any $x \leq \underline{z}$ and $v = \underline{z} + \varepsilon$ where $\varepsilon > 0$ is small. Therefore it is optimal for the firm to attract workers by posting $v > \underline{z}$. Since

$$\Pi(v; v_{t-1}) = \mathcal{M}(v)\phi(v; v_{t-1}) + \beta\Pi(v) = \mathcal{M}(v)\phi(v; v_{t-1}) + \beta\Pi(v^*)$$

for any $v \in (\underline{z}, v^*)$, it is clear that v_t^* can be uniquely found by the strict concavity of $\mathcal{M}(v)\phi(v; v_{t-1})$ for $v \in (\underline{z}, v^*)$. Therefore in period t, the firm uniquely posts $v_t^* \in (\underline{z}, v^*)$. In subsequent periods, by Theorem 1, the firm posts v^*. $\square$

Lemma 8 says that when $v_{t-1} \in (\bar{v}, \bar{z})$, the firm will overreact by immediately lowering the wage *below* the target level v^*, and then bringing it back up to v^* the period after. Intuitively, the overreaction occurs because large adjustment cost have a very strong impact on the intensive margin effect of wage cuts. To see this, note that in this equilibrium the adjustment cost is very large, all else equal. Decreasing labor costs is thus of primary importance for the firm as compared to raising the probability of match with a worker. It follows that it is optimal for the firm to manage large adjustment costs by sharply reducing labor costs, posting a wage below the target v^*. Next period, the firm can costlessly raise the wage back to the target level to optimally compete for workers with other firms.

The next lemma considers the case when yesterday's wage is extremely off target, i.e., $v_{t-1} \geq \bar{z}$.

Lemma 9. *Let $v_{t-1} \geq \bar{z}$ on date t. The firm will be inactive in period t and re-enter it in period $t+1$, optimally setting $v^*_{t+j} = v^*$ for all $j \geq 1$.*

The proof immediately follows from the earlier observation that if $v_{t-1} \geq \bar{z}$, then the firm cannot achieve a positive payoff in t due to the large adjustment cost. This effectively forces the firm off the market in t, in order to re-enter it in $t+1$.

3.4 Equilibrium wages after a permanent shock

Here we apply the lessons from Theorem 1 to the case of an unanticipated permanent shock that moves the firm's break-even wage $\bar{v}$ to a new level $\bar{\omega}$. As a consequence, the target wage will change from v^*, to a new level denoted ω^*. We will differentiate between positive shocks in which $\bar{\omega} > \bar{v}$, and negative shocks, where $\bar{\omega} < \bar{v}$.[5]

Equilibrium wages will respond asymmetrically, depending on whether the unanticipated shock is positive or negative. In particular, wages can be "downward sticky." The firm will immediately raise wages to the new target $\omega^* > v^*$ in response to a positive shock. But, it will react differently when the shock is negative. A firm that is subject to a small negative shock might choose not to bring wages down to the new target ω^*, keeping them at the old target v^* indefinitely. Instead, a large negative shock may induce the firm to partially or fully lower wages to the new target ω^*, depending on the size of the shock. A very large negative shock will force the firm to leave the market temporarily or permanently.

To show this, we only need to consider shocks that reduce the firm's break-even wage so much that the new break-even level ends up being below yesterday's wage, $\bar{\omega} < v_{t-1}$. In this case only wages below $\bar{\omega}$ guarantee a positive profit to the firm on t; hence, since $\bar{\omega} < v_{t-1}$, a firm who wants to remain in the market will have to lower wages relative to what it posted the day before.

[5]The source of the shock is not important at this point, although later in this Section we will provide some examples.

It is immediate that if the negative shock is so large that the new break even wage falls below the exit wage, then the firm will permanently exit the market since it can no longer profitably hire workers.

Therefore, consider the case when the negative shock is not so extreme. That is, the break-even wage $\bar{\omega}$ remains above the exit wage. Following Theorem 1, there can be two cases. In the more favorable case, the firm will overreact by immediately lowering the wage *below* the new target level, raising the wage to the target level only tomorrow. In the less favorable case, the firm exists the market and re-enters it next period by posting the new target wage $v^*_{t+1} = \omega^*$.

It is worth mentioning that the analysis carries over to the case of market-wide shocks in which a small measurable set of firms receives a shock that does not change the exit wage. Shocks of this type might change the demand for each firm if the shocks induce some firms to exit the market and they may alter the workers' market utility U whenever firms adjust wages or exit the market. In all of these cases each firm will have a new target wage ω^* and the preceding analysis can be adapted to study wage stickiness. Some examples of this kind are offered in the Appendix.

4 Discussion

This paper has extended the basic directed search model by introducing a variable wage adjustment cost. The set-up considers dynamic, two-sided labor markets where matching frictions *and* downward wage rigidity emerge as an equilibrium phenomenon. In the model firms compete for workers by posting wages and the distribution of demand for jobs is endogenous because workers choose where to apply. This basic directed search model is modified by introducing a variable cost that a firm incurs when it wants to adjust the wage downward relative to the past wage. The analysis identifies four endogenous wage thresholds, which allow a full characterization of equilibrium wage stickiness: an *exit* wage, below which the firm cannot recruit workers; a *target* wage, which maximizes the firm's profit in the absence of any adjustment cost from lowering wages; a *trigger* wage, below which the firm optimally *chooses* not to respond to unanticipated shocks; a *break-even* wage, above which there is no wage rigidity; and an *inactive threshold* wage above which the firm prefers to remain temporarily idle.

The study complements other studies of wage rigidity, which have been conducted using one-sided labor markets where frictions are due to costs from hiring or searching for appropriate counterparts. One approach is to directly assume a suitable real wage schedule (e.g., Blanchard and Galí, 2010). By contrast, in our model the underlying wage selection mechanism is made explicit. Another approach is to leverage the idea of focal points in bargaining when wage strategies are history- and state-independent (Hall, 2005). By contrast, our analysis does not rely on assuming that players are able to coordinate on a market-wide scale. A third approach revolves around the idea that if market participants can commit to long-term plans, then insurance motives may lead to a constant wage for risk-averse workers (Rudanko, 2009). By contrast, commitment is relaxed in our analysis.

The argument made in the paper relies on the assumption that the adjustment cost linearly depends on the magnitude of the wage cut, and does not depend on the level of the old wage. An alternative formulation of nonlinear adjustment costs could alter the main result, but only when differences between the target wage and the previous period's wage are sufficiently large. For sufficiently small differences   differences that imply a small variation in the optimal wage v^*, that is   then our linear formulation would still be a good approximation for any smooth cost function, so nonlinearities would not alter the result. However, for large differences this approximation would not be suitable and this could have consequences. In particular, a convex cost function might remove the incentive to perform large wage cuts. If so, then the firm would find it optimal to perform a sequence of small wages cuts, instead of one single, large wage cut.

Our directed search model has the key characteristic that the matching function is endogenous, players are searching in each period and cannot commit to future wages or matches. It is therefore instructive to consider these features in relation to the literature where the random matching function is exogenous (e.g. urn-ball matching), and where matches can be multi-period with external separation shocks.

On the first point, whether search is directed or random matters when there is a wage adjustment cost. Our analysis cannot be simplified by imposing an urn-ball matching process (e.g., Stevens, 2007). The micro-foundation for the matching function $\mathcal{M}$ is not equivalent to an urn-ball matching specification. In the latter, wage changes cannot increase the firm's matching probability because meetings are exogenously random. By contrast, in our model search is endogenously random and wage changes affect the firm's matching rate in and out of equilibrium. In particular, offering a wage below competitors lowers the probability to meet workers, even in an infinitely large market. Consequently, endogenizing the matching function has implications for firms' desire to cut wages following a shock. With random matching, a firm who has variable adjustment costs adjusts the wage if it lays above the reservation value. This is not necessarily true with directed search, because workers will react by decreasing the probability to visit the firm. This means a shorter queue for the firm, hence lower expected profit. This tradeoff is absent under random matching, which therefore makes wage stickiness less likely, even under variable adjustment costs.

On the second point, the analysis suggests that, if we relaxed the assumption of 100% separation rate at the end of each period, then there would be scope for wage rigidity of existing workers versus new hires. To understand why we think this would be so, for the sake of the argument, consider a firm-worker pair in which the firm hired the worker yesterday. Suppose that the match can be sustained for as long as it is in the best interest of the two parties. What would the firm pay the worker today? One can prove that both worker and firm have an interest in remaining matched as long as the wage does not drop below the threshold represented by the market utility

U. The worker has no incentive to leave the firm and search for another job if the wage $v \geq U$; and the firm has no interest in paying her more than U. So $v = U$ is optimal. The wage of an existing worker would thus be "sticky" at U, as long as the firm does not suffer a shock forcing it out of the market at that wage. However, this would not be so if the firm had to advertise for new hires, because the offered wage would vary with the firm's shocks. In this sense, our model is consistent with the empirical observation that wage rigidity is more prominent for pre-existing workers, but not for new hires (Haefke et al., 2013).

References

Agell, J., and P. Lundborg (2003). Survey evidence on wage rigidity and unemployment. *Scandinavian Journal of Economics*, 105, 15 30.

Albrecht, J., Gautier, P. A., and S. Vroman (2006). Equilibrium directed search with multiple applications. *Review of Economic Studies*, 73(4), 869 891.

Ball, L., and D. Romer (1991). Sticky prices as coordination failure. *The American Economic Review*, 81(3), 539 552.

Bewley, T. F. (1995) A depressed labor market as explained by participants. *The American Economic Review*, 85(2), 250 254.

Blanchard, O.J., and J. Galí (2010). Labor markets and monetary policy: a new Keynesian model with unemployment. *American Economic Journal: Macroeconomics*, 2(2), 1 30.

Burdett, K., Shi, S., and R. Wright (2001). Pricing and matching with frictions. *Journal of Political Economy*, 109(5), 1060 1085.

Camera, G., and J. Kim (2016). Dynamic directed search. *Economic Theory*, 62(1), 131 154.

Campbell III, C. M., and K. S. Kamlani (1997). The reasons for wage rigidity: evidence from a survey of firms. *The Quarterly Journal of Economics*, 112(3), 759 789.

Fehr, E., and A. Falk (1999). Wage rigidity in a competitive incomplete contract market. *Journal of Political Economy*, 107(1), 106 134.

Haefke, C., Sonntag, M., and T. van Rens (2013). Wage rigidity and job creation. *Journal of Monetary Economics*, 60(8), 887 899.

Hall, R. E. (2005). Employment fluctuations with equilibrium wage stickiness. *American Economic Review*, 95(1), 50 65.

Holzer, H. J., and E. B. Montgomery (1993). Asymmetries and rigidities in wage adjustments by firms. *The Review of Economics and Statistics*, 75(3), 397 408.

Kaufman, R. T. (1984). On wage stickiness in Britain's competitive sector. *British Journal of Industrial Relations*, 22(1), 101 112.

Kim, J., and G. Camera (2014). Uniqueness of equilibrium in directed search models. *Journal of Economic Theory*, 151, 248 267.

Kube, S., Maréchal, M. A., and C. Puppe (2013). Do wage cuts damage work morale? Evidence from a natural field experiment. *Journal of the European Economic Association*, 11(4), 853 870.

Norman, P. (2015). Matching with frictions and entry with Poisson distributed buyers and sellers. *Manuscript*, University of North Carolina.

Peters, M. (1984). Bertrand equilibrium with capacity constraints and restricted mobility. *Econometrica*, 52(5), 1117 1127.

Rudanko, L. (2009). Labor market dynamics under long-term wage contracting. *Journal of Monetary Economics*, 56, 170 183.

Shapiro, C., and J. E. Stiglitz (1984). Equilibrium unemployment as a worker discipline device. *The American Economic Review*, 74(3), 433 444.

Shimer, R. (2005). The assignment of workers to jobs in an economy with coordination frictions. *Journal of Political Economy*, 113(5), 996 1025.

Stevens, M. (2007). New microfoundations for the aggregate matching function. *International Economic Review*, 48(3), 847 868.

Virág, G. (2010). High profit equilibria in directed search models. *Games and Economic Behavior*, 71, 224 234.

Appendix

Derivation of $\frac{\partial\lambda}{\partial v}$ and $\frac{\partial^2\lambda}{\partial v^2}$.

Given $U > 0$, λ is a function of v such that $H(\lambda)v = U$. i.e. $(1-e^{-\lambda})v = U\lambda$. Hence

$$e^{-\lambda}v\frac{\partial\lambda}{\partial v} + 1 - e^{-\lambda} = U\frac{\partial\lambda}{\partial v} \qquad \Rightarrow \qquad \frac{\partial\lambda}{\partial v} = \frac{1-e^{-\lambda}}{U - e^{-\lambda}v}.$$

By substituting $U = \frac{1-e^{-\lambda}}{\lambda}v$, we have

$$\frac{\partial\lambda}{\partial v} = \frac{\left(1-e^{-\lambda}\right)^2}{U\left[1-(1+\lambda)e^{-\lambda}\right]} > 0.$$

Hence

$$\begin{aligned}
\left[1-(1+\lambda)e^{-\lambda}\right]^2 U\frac{\partial^2\lambda}{\partial v^2} &= \frac{\partial\lambda}{\partial v}\left\{2\left(1-e^{-\lambda}\right)e^{-\lambda}\left[1-(1+\lambda)e^{-\lambda}\right] - \left(1-e^{-\lambda}\right)^2\lambda e^{-\lambda}\right\} \\
&= \frac{\partial\lambda}{\partial v}\left(1-e^{-\lambda}\right)e^{-\lambda}\left\{2\left[1-(1+\lambda)e^{-\lambda}\right] - \lambda\left(1-e^{-\lambda}\right)\right\} \\
&= \frac{\partial\lambda}{\partial v}\left(1-e^{-\lambda}\right)e^{-\lambda}\left[2-\lambda-(\lambda+2)e^{-\lambda}\right].
\end{aligned}$$

Let $g(\lambda) = 2 - \lambda - (\lambda+2)e^{-\lambda}$. Then $g(0) = 0$ and

$$g'(\lambda) = -1 + (1+\lambda)e^{-\lambda} < 0 \qquad \text{for all } \lambda > 0.$$

Hence $g(\lambda) < 0$ for all $\lambda > 0$. As a result, we have

$$\frac{\partial^2\lambda}{\partial v^2} < 0.$$

Examples

In this section we provide two examples of possible shocks that alter the firm's target wage. A first possibility is that the firm experiences a productivity shock, which changes the underlying cost structure of the firm. For example, consider a shock that results in a parallel shift of the profit function ϕ.

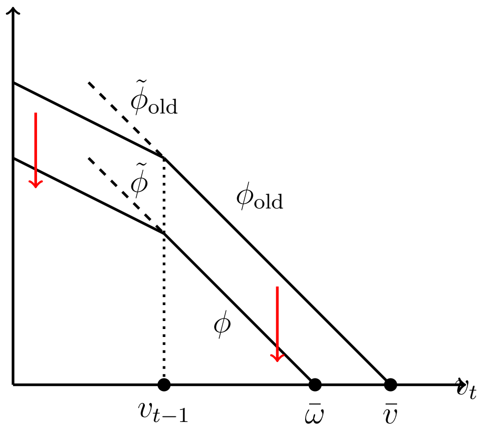

Figure 8: The impact of a small negative shock the profit function ϕ_{old}.

Figures 8 illustrates a small negative productivity shock on period t, which lowers the break-even wage to $\bar{\omega} < \bar{v}$.

We can also consider unanticipated demand shocks. Let $\dfrac{|\mathcal{I}|}{|\mathcal{J}|} = r$ be the new market queue following a shock on date t. This change in the queue alters the target wage. To find the new target wage v^*, we need to find the solution to

$$\arg\max_{v} \mathcal{M}(\lambda; r)\tilde{\phi}(v)$$

such that $\mathcal{H}(\lambda(v; r))v = \mathcal{H}(\lambda_j(v_j; r))v_j$ for any $j \in \mathcal{J}$.

As the market queue r increases, the target wage v^* decreases. i.e.

$$\frac{\partial v^*}{\partial r} < 0.$$

To see this note that the first order condition at the target wage v^* is

$$\frac{\partial \mathcal{M}(\lambda(v^*; r))}{\partial \lambda} \frac{\partial \lambda(v^*; r)}{\partial v}(\bar{v} - v^*) - \mathcal{M}(\lambda(v^*; r)) = 0.$$

By the implicit function theorem,

$$\frac{\partial v^*}{\partial r} = -\frac{C(v^*; r)}{B(v^*; r)}$$

where

$$C(v^*; r) := \left(\frac{\partial^2 \mathcal{M}(\lambda(v^*;r))}{\partial \lambda^2} \frac{\partial \lambda(v^*;r)}{\partial v} \frac{\partial \lambda(v^*;r)}{\partial r} + \frac{\partial \mathcal{M}(\lambda(v^*;r))}{\partial \lambda} \frac{\partial^2 \lambda(v^*;r)}{\partial r \partial v} \right)(\bar{v} - v^*) - \frac{\partial \mathcal{M}(\lambda(v^*;r))}{\partial \lambda} \frac{\partial \lambda(v^*;r)}{\partial r} < 0$$
$$B(v^*; r) := \left\{ \frac{\partial^2 \mathcal{M}(\lambda(v^*;r))}{\partial \lambda^2} \left(\frac{\partial \lambda(v^*;r)}{\partial v} \right)^2 + \frac{\partial \mathcal{M}(\lambda(v^*;r))}{\partial \lambda} \frac{\partial^2 \lambda(v^*;r)}{\partial v^2} \right\}(\bar{v} - v^*) - 2\frac{\partial \mathcal{M}(\lambda(v^*;r))}{\partial \lambda} \frac{\partial \lambda(v^*;r)}{\partial v} < 0$$

$B(v^*; r)$ represents the second order condition. It is also easy to see $C(v^*; r) < 0$.

Given v, as the market queue r increases, the demand for the each firm increases, i.e. $\frac{\partial \lambda(v;r)}{\partial r} > 0$, so it is straightforward to see that the last term of $C(v^*;r)$ is negative. i.e. $-\frac{\partial \mathcal{M}(\lambda(v^*;r))}{\partial \lambda}\frac{\partial \lambda(v^*;r)}{\partial v}\frac{\partial \lambda}{\partial r} < 0$. For the remaining terms, we have

$$\begin{aligned}
&\frac{\partial^2 \mathcal{M}(\lambda(v^*;r))}{\partial \lambda^2}\frac{\partial \lambda(v^*;r)}{\partial v}\frac{\partial \lambda}{\partial r} + \frac{\partial \mathcal{M}(\lambda(v^*;r))}{\partial \lambda}\frac{\partial^2 \lambda(v^*;r)}{\partial r \partial v} \\
&\qquad = -e^{-\lambda}\frac{\lambda(1-e^{-\lambda})}{v[1-(1+\lambda)e^{-\lambda}]}\frac{\partial \lambda}{\partial r} + e^{-\lambda}\frac{(1-e^{-\lambda})^2 - \lambda^2 e^{-\lambda}}{v[1-(1+\lambda)e^{-\lambda}]^2}\frac{\partial \lambda}{\partial r} \\
&\qquad = \frac{-e^{-\lambda}}{v[1-(1+\lambda)e^{-\lambda}]^2}\frac{\partial \lambda}{\partial r}[\lambda^2 e^{-2\lambda} - (1-\lambda)(e^{-\lambda}-1)^2] < 0.
\end{aligned}$$

We obtain the second line using $\frac{\partial \lambda(v^*;r)}{\partial v} = \frac{\lambda(1-e^{-\lambda})}{v[1-(1+\lambda)e^{-\lambda}]}$. Therefore $\frac{\partial v^*}{\partial r} < 0$.