3.2 Super-oscillations

Superoscillations are functions which oscillate with an arbitrarily high frequency α, but which, surprisingly, can be understood as superpositions of low frequencies, $|k| < 1$, seemingly a violation of the Fourier theorem:

$$\sum_{|k|<1} c_k e^{ikx} \to e^{i\alpha x} \tag{3.16}$$

Superoscillations were originally discovered through the study of weak-values. By way of example:

$$\begin{aligned}
|\Phi^{MD}_{fin}\rangle &= \left\{\cos\frac{\lambda\hat{Q}_{md}}{N} - i\alpha_w \sin\frac{\lambda\hat{Q}_{md}}{N}\right\}^N |\Phi^{\text{MD}}_{\text{in}}\rangle \\
&= \left\{\frac{e^{\frac{i\lambda\hat{Q}_{md}}{N}} + e^{-\frac{i\lambda\hat{Q}_{md}}{N}}}{2} + \alpha_w \frac{e^{\frac{i\lambda\hat{Q}_{md}}{N}} - e^{-\frac{i\lambda\hat{Q}_{md}}{N}}}{2}\right\}^N |\Phi^{\text{MD}}_{\text{in}}\rangle \\
&= \underbrace{\left\{e^{\frac{i\lambda\hat{Q}_{md}}{N}}\frac{(1+\alpha_w)}{2} + e^{-\frac{i\lambda\hat{Q}_{md}}{N}}\frac{(1-\alpha_w)}{2}\right\}^N}_{\equiv\psi(x)} |\Phi^{\text{MD}}_{\text{in}}\rangle
\end{aligned} \tag{3.17}$$

We already saw how this could be approximated as $e^{i\lambda\alpha_w\hat{Q}_{md}}|\Phi^{\text{MD}}_{\text{in}}\rangle$ which produced a robust-shift in the measuring-device by the weak-value $\sqrt{2}$. However, we can also view $\psi(x) = \left\{e^{\frac{i\lambda\hat{Q}_{md}}{N}}\frac{(1+\alpha_w)}{2} + e^{-\frac{i\lambda\hat{Q}_{md}}{N}}\frac{(1-\alpha_w)}{2}\right\}^N$ in a very different way, by performing a binomial expansion:

$$\begin{aligned}
\psi(x) &= \sum_{n=0}^{N} \frac{(1+\alpha_w)^n(1-\alpha_w)^{N-n}}{2^N}\frac{N!}{n!(N-n)!}\exp\left\{\frac{in\lambda\hat{Q}_{\text{md}}}{N}\right\}\exp\left\{\frac{-i\lambda\hat{Q}_{\text{md}}(N-n)}{N}\right\} \\
&= \sum_{n=0}^{N} c_n \exp\left\{\frac{i\lambda\hat{Q}_{\text{md}}(2n-N)}{N}\right\} = \sum_{n=0}^{N} c_n \exp\left\{\frac{i\lambda\hat{Q}_{\text{md}}\lambda_n}{N}\right\}
\end{aligned} \tag{3.18}$$

Because these regions of superoscillations are created at the expense of having the function grow exponentially in other regions, it would be natural to conclude that the superoscillations would be quickly "over-taken" by tails coming from the exponential regions and would thus be short-lived. However, it has been shown that superoscillations are remarkably robust and can last for a surprisingly long time. This has therefore led to proposed/practical applications of superoscillations to situations which were previously probed by evanescent waves (e.g. in the superresolution of very fine features with lasers).