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# *The Philosophical Significance of*
# Gödel's Dialectica Translation

*Abstract.* Hilbert's Program in the 1920s aimed to give finitary consistency proofs for infinitary mathematics, thus putting infinitary mathematics on a more secure footing. There is a popular narrative that Hilbert's Program was decisively refuted by Gödel's incompleteness theorems in 1931. However, Hilbert's school continued to work on consistency proofs for decades after the appearance of the incompleteness theorems. Moreover, Gödel himself, in a remarkable paper of 1958, pursues a modified version of Hilbert's Program: he presents his Dialectica translation as a new, Hilbert-style consistency proof for arithmetic based on "an extension of the finitary standpoint," and he clearly regards this proof as epistemologically significant. So, is there any truth to the claim that Hilbert's Program was refuted by the incompleteness theorems? What epistemological significance did Gödel ascribe to his Dialectica translation, and was he correct about this?

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|---|---|
| 1925 | Hilbert's lecture "On the infinite" |
| 1931 | G's incompleteness theorems |
| 1941 | G presents the Dialectica translation at Princeton and Yale |
| 1958 | G publishes the Dialectica translation (in German) |
| 1967-72 | G works on a revised, expanded English translation of the 1958 paper |

## §1. Hilbert's Program after incompleteness

*Hilbert's Program:* to give a finitary consistency proof for infinitary mathematics.

*Problem of the infinite:* actual infinity is not representable in concrete intuition.
So, reasoning about the actual infinite is epistemologically problematic. (?)

*Characterizations of finitary mathematics:*

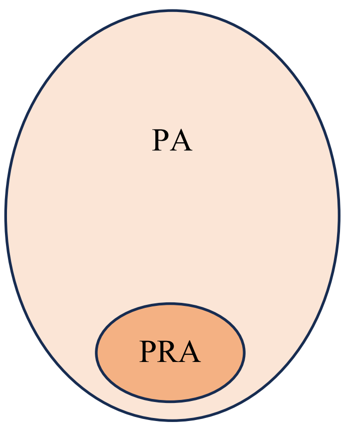
- based on thought experiments conducted in concrete intuition
- more "secure" or "evident" than infinitary mathematics

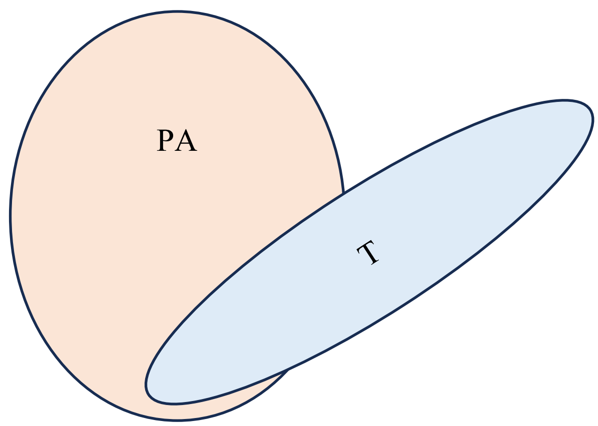
- no existential quantifiers ranging over an infinite totality
- e.g., “2+3 = 5” and “x+y = y+x” are finitary statements, but “there is a prime number greater than 10^(10^100)” and “for every number x, there is a prime p > x” are infinitary.

A *consistency proof* would establish the *finitary correctness* of infinitary mathematics: i.e., every finitary statement provable using infinitary methods is true. This would provide an *instrumentalist* justification for the use of infinitary mathematics (against Kronecker, Brouwer, Weyl).

*Second incompleteness theorem:* for any sufficiently powerful mathematical theory U, there can be no consistency proof for U using only the means of proof available within U itself.

It was widely agreed that the methods of Hilbert’s finitary mathematics would be included within those of PA. So, Hilbert’s Program fails?




*Hilbert’s Program 2.0:* to give a constructive consistency proof for non-constructive mathematics.

- What does “constructive” mean?
- Why is constructive math epistemologically better than non-constructive math?

I think:

- G has a principled definition of “constructive,” based on an interesting analysis of Hilbert’s finitism.
- But he has no good answer to the second question.
- The real philosophical significance of the Dialectica translation lies in *assigning a constructive meaning* to arithmetic (not as a consistency proof).

## §2. The Dialectica paper

*Gödel 1972, "On an extension of finitary mathematics that has not yet been used" (revised English translation of Gödel 1958)*

P. Bernays has pointed out on several occasions that, in view of the fact that the consistency of a formal system cannot be proved by any deduction procedures available in the system itself, it is necessary to go beyond the framework of finitary mathematics in Hilbert's sense in order to prove the consistency of classical mathematics or even of classical number theory. Since finitary mathematics is defined as the mathematics of *concrete intuition*, this seems to imply that *abstract concepts* are needed for the proof of consistency of number theory. An extension of finitism by such concepts was explicitly suggested by Bernays in his *1935*, page 69. By abstract concepts, in this context, are meant concepts which are essentially of the second or higher level, i.e., which do not have as their content properties or relations of *concrete objects* (such as combinations of symbols), but rather of *thought structures* or *thought contents* (e.g., proofs, meaningful propositions, and so on), where in the proofs of propositions about these mental objects insights are needed which are not derived from a reflection upon the combinatorial (space-time) properties of the symbols representing them, but rather from a reflection upon the *meanings* involved.

[…]

At any rate Bernays' observations in his *1935*, footnote 1, teach us to distinguish two component parts in the concept of finitary mathematics, namely: first, the *constructivistic* element, which consists in admitting reference to mathematical objects or facts only in the sense that they can be exhibited, or obtained by construction or proof; second, the specifically *finitistic* element, which requires in addition that the objects and facts considered should be given in concrete mathematical intuition. This, as far as the objects are concerned, means that they must be finite space-time configurations of elements whose nature is irrelevant except for equality or difference. (In contrast to this, the objects in intuitionistic logic are meaningful propositions and proofs.)

It is the second requirement which must be dropped. Until now this fact was taken into account by adjoining to finitary mathematics parts of intuitionistic logic and of the constructivistic theory of ordinal numbers. It will be shown in the sequel that, instead, one can use, for the proof of consistency of number theory, a certain concept of a *computable function of finite type* over the natural numbers and some very elementary axioms and principles of construction for such functions.

Comments:

- G presents his contribution modestly, as a development within Hilbertian tradition.
- But for HB, finitary math is more secure than infinitary math *because* it is rooted in concrete intuition. So, G is really throwing HB's philosophical framing out the window.

**§3. What is the "constructivistic element"?**

*Gödel 1941, Yale lecture (CW vol. 3, p. 191)*

> Let me call a system strictly constructive or finitistic if it satisfies these three requirements (relations and functions decidable, respectively, calculable, no existential quantifiers at all, and no propositional operations applied to universal propositions). I don't know if the name "finitistic" is very well chosen, but there is certainly a close relationship between these systems and what Hilbert called the "finite Einstellung".

I think that G's "constructivistic element" is a convoluted way of saying: no existential quantifiers, and no negated universal quantifiers. Such quantifiers are a way of "admitting reference" to mathematical objects in a sense other than actually exhibiting them (i.e., naming them explicitly).

*Gödel 1941, Princeton lectures (published 2021, eds. Hämeen-Anttila and von Plato, p. 32)*

> the problem arises how to axiomatize mathematics in such a manner that such undesirable things as non-constructive existence proofs can never happen i.e. such that the proof of any existential proposition yields a way to find the thing whose existence is asserted.

Comments:

- The "problem of the infinite" disappears from G's extended finitism; it is replaced by the problem of *avoiding non-constructive existence proofs* (but see Appendix).
- Non-constructive existence proofs are indeed strangely uninformative!

**§4. The problem of getting more certainty**

G claims that we can get more certainty by reducing non-constructive to constructive systems.

*Gödel 1938, Lecture at Zilsel's (Collected Works vol. 3, p. 89)*

> The question [of a consistency proof] has, however, also an epistemological side. *After all we want a consistency proof for the purpose of a better foundation of mathematics (laying the foundations more securely)*, and there can be mathematically very interesting proofs that

do not accomplish that (as, for example, Tarski's for analysis). A proof is only satisfying if it either

A. *reduces to a proper part* or
B. *reduces to something which, while not a part, is more evident, reliable, etc., so that one's conviction is thereby strengthened.*

A signifies without doubt an objective step forward (making assumptions superfluous is almost the same as a proof). B is at the outset problematic because subjectively different—but de facto not so bad, since there exists general agreement that constructive systems are better than those that work with the existential "there is". And also historically the task has been to reduce non-constructive to constructive mathematics.

Comments:

- Incompleteness complicates the philosophical task by ruling out the possibility of A.
- B is problematic. Practically everyone is convinced that PA is consistent.
- Any constructivistic system capable of proving the consistency of PA will be just as combinatorially complicated as PA.
- G describes non-constructive existence proofs as "strange," "surprising," "undesirable." But he never explains why such proofs should be less "evident" or "reliable" than constructive proofs.

## §5. What is the Dialectica translation?

The Dialectica translation is an interpretation* of arithmetic** in a quantifier-free*** theory T of computable functions of finite type****.

*roughly, a translation that carries theorems to theorems
**intuitionistic first-order arithmetic (HA); can also get classical first-order arithmetic (PA)
***free of existential quantifiers and negated universal quantifiers
****generalization of primitive recursion to higher types; easily defines Wainer hierarchy ("iterated iteration")

To each arithmetical formula $A(z)$, Gödel assigns a formula $A_D(x, y, z)$ in the language of T such that:

- $A(z)$ is "almost" logically equivalent to $\exists x\, \forall y\, A_D(x, y, z)$,
- If HA proves $A(z)$, then T proves $A_D(Q(z), y, z)$ for some constants Q that we can actually exhibit.

Comments:

- Interpretation yields relative consistency: if T is consistent, then so is PA (provably in PRA).

- Assertions in T essentially have the form f(x,y,z,…) = g(x,y,z,…), where f,g are explicitly described computable functions of finite type.
- The meaning of $A(z)$ is very close to that of $\exists x\, \forall y\, A_D(x, y, z)$. The Dialectica translation is just a more constructive version of Skolemization.
- *Circularity objection:* does G's definition of "computable function of type t" presuppose quantificational logic?

## §6. What Gödel should have said

*Brouwer's objection:* mere consistency is not enough to ensure that a theory is meaningful and true. (In other words, Brouwer did not accept Hilbert's instrumentalist gambit!)

> *Brouwer 1925, in van Heijenoort, p. 336*
>
> We need by no means despair of reaching this goal [viz., that of giving a consistency proof for classical infinitary mathematics], but nothing of mathematical value will thus be gained: an incorrect theory, even if it cannot be inhibited by any contradiction that would refute it, is none the less incorrect, just as a criminal policy is none the less criminal even if it cannot be inhibited by any court that would curb it.

The Dialectica translation is exactly what we need to meet Brouwer's objection! For it assigns a constructive meaning to quantificational statements.

- The constructive meaning is in general not a statement but a *problem:* to find the witnessing constants Q. For theorems of PA, we can solve the problem.
- The Dialectica translation shows how to systematically extract witnessing constants from any proof in PA—including non-constructive existence proofs!
- Caveat: these constants may be of higher type.

## §7. Conclusions

The consistency of T is no more certain or evident than that of PA.

But the Dialectica translation achieves a *deeper reconciliation* between classical and constructive mathematics than the one that Hilbert sought: it assigns a constructive meaning to classical mathematics. Moreover, the interpretation is cogently connected with the intended infinitary meaning.

Constructivism can be *enriching* and not merely subtractive!

## Appendix

*Fraenkel 1925, "Zehn Vorlesungen über die Grundlegung der Mengenlehre," pp. 36-37*

**The fundamental thesis: mathematical existence = constructability.** The fundamental intuition, from which all the assertions of the intuitionists (some of which seem so surprising) can be derived more or less logically, concerns a point already touched upon in a special sense in No. 5 above: *the sharp distinction between constructions and pure existence statements, and the sole recognition of the former while rejecting the latter*. There are proofs for the existence of certain mathematical notions (numbers, functions, sets etc.) in the most diverse areas of modern mathematics, even within arithmetic, which do not demonstrate this existence through step-by-step constructive production out of simpler notions, but rather through the use of a step that is not constructively resolvable; e.g. by proving that the non-existence of the notion in question stands in contradiction with proven theorems or accepted principles, yet without this contradiction giving a way of producing the notion. Such a proof of course (in contrast with constructive proofs) allows no closer insight into the nature of the notion in question; if, for example, the existential statement only expresses that a constant of a certain meaning is a finite integer, then the existence proof gives us no handle on determining the size of this integer. Proofs of this kind have hitherto been not only accepted, but even especially prized and admired on account of the great ingenuity which they usually require. (A classic example is HILBERT'S first, "theological" proof for the existence of finite invariant systems; compare lects. 5/6, end of 7.)

*N.B.,* Fraenkel uses "intuitionists" very broadly to encompass Kronecker, Borel, Lebesgue, Poincare, Brouwer, Weyl (pp. 34-35).

*Gödel 1972, footnote* **b**

"*Concrete intuition*", "*concretely intuitive*" are used as translations of "Anschauung", "anschaulich". The simple terms "*concrete*" or "*intuitive*" are also used in this sense in the present paper. What Hilbert means by "Anschauung" is substantially Kant's space-time intuition confined, however, to configurations of a finite number of discrete objects. Note that it is Hilbert's insistence on *concrete* knowledge that makes finitary mathematics so surprisingly weak and excludes many things that are just as incontrovertibly evident to everybody as finitary number theory. E.g., while any primitive recursive definition is finitary, the general principle of primitive recursive definition is not a finitary proposition, because it contains the abstract concept of function. There is nothing in the term "finitary" that would suggest a restriction to concrete knowledge. Only Hilbert's special interpretation of it introduces this restriction.