8th International Meeting of the Association for the Philosophy of Mathematical Practice (APMP)

Chapman University

January 12-15, 2026

ABSTRACTS

of Contributed Talks

Andrew Aberdein (Florida Institute of Technology). *Mathematical Trespassing.*

Abstract. "Epistemic trespassing" was proposed by Nathan Ballantyne as a characterization of the behaviour of experts who make pronouncements outside the domain of their expertise [2]. Such trespassing can often be a productive exercise, but caution is required: Ballantyne warns trespassers that they should substantially reduce the confidence of their assertions in the new domain unless they have acquired cross-field expertise, whether directly by training or vicariously through collaboration.

This paper applies epistemic trespassing to mathematics. I maintain that mathematical practice is rich with examples of epistemic trespassing. I distinguish three general varieties: internal (or intramathematical) trespassing between subfields of mathematics; inbound extramathematical trespassing from other disciplines into mathematics; and outbound extramathematical trespassing from mathematics into other disciplines. I further distinguish each variety into benign and malign trespassing and discuss examples of each.

Internal trespassing done right can be a powerful source of creative insight. Some celebrated mathematicians, such as Paul Erdős and William Thurston, are noted for their ability to successfully apply methods drawn from apparently unrelated areas. But internal trespassing done wrong may represent an obstacle to mathematical progress. Troubling cases include Shinichi Mochizuki's characterization of the critics of his claimed proof of the abc conjecture as unqualified and some appeals to methodological purity.

Bad inbound trespassers are familiar as cranks: some of them are experts in their own fields but painfully out of their depth in mathematics [3]. But there are also good inbound trespassers: typically, as Ballantyne suggests, individuals who have done the work to acquire relevant expertise. A notable example is the lay mathematician Marjorie Rice's work on tiling [4].

Outbound trespassing presents some of the most interesting examples. In the broadest sense, applied mathematics, at least as developed by mathematicians, is all outbound trespassing. Much of it, of course, is highly successful, but there are also examples of failed or inappropriate applications. Some responses to the unreasonable effectiveness puzzle and accounts of mathematical explanations in science (MES) indicate possible defences against bad outbound trespassing [5;1]. On the negative side, some public policy debates appear to function as "attractive nuisances", or incitements to trespass (examples include Serge Lang's publications on the relationship between HIV and AIDS). I also discuss a possible example of "counter-trespassing": defensible outbound trespassing in a controversial area that provoked less legitimate outbound trespassing by other mathematicians in reply.

These examples help to qualify some features of Ballantyne's account of epistemic trespassing and also serve to unify some frequently (and other less frequently) discussed areas of mathematical practice.

References

[1] Baker, Alan. 2012. Science-driven mathematical explanation. Mind 121:243–267.

[2] Ballantyne, Nathan. 2019. Epistemic trespassing. Mind 128:367–395.

[3] Rittberg, Colin Jakob. 2023. Justified epistemic exclusions in mathematics. Philosophia Mathematica 31:330–359.

[4] Schattschneider, Doris. 2018. Marjorie Rice and the MAA tiling. Journal of Mathematics and the Arts 12:114–127.

[5] Waxman, Daniel. 2021. Is mathematics unreasonably effective? Australasian Journal of Philosophy 99:83–99.

Keywords: applied mathematics, cranks, epistemic trespassing

Marianna Antonutti Marfori (IHPST/Université Paris 1 Panthéon-Sorbonne). ***Frege and Hilbert on mathematical method and proof.***

Abstract. The disagreement between Frege and Hilbert, expressed especially in their correspondence, is chiefly concerned with the nature of axioms and their implication for mathematical existence. This disagreement has deep epistemological and semantic consequences for their views of logic and mathematics, including their respective conceptions of consistency and independence proofs, and more generally the proper method of mathematics (axiomatic vs. genetic). In particular, much debate has focused on the reasons why the question of consistency proofs, which is at the core of Hilbert's foundational programme, does not arise in the context of Frege's philosophy of mathematics. In this talk, I will argue that despite their disagreement on the nature of axioms, their positions concerning mathematical method and proof are close in certain important respects which have significantly shaped the way in which we think of mathematical proofs. This consideration is motivated by the importance that the notion of a rigorous proof plays in both Frege's and Hilbert's work. While they had very different views on logic and its relationship to mathematics, both viewed ideally rigorous proofs as formal, gapless arguments in which all the underlying assumptions must be made explicit, and the inference steps must be carried out according to specified rules, in order to prevent implicit reliance on intuition at any point in a mathematical proof. For both, the justification of a mathematical statement must rely on the logical structure of the proof only. One way in which Frege's and Hilbert's different conceptions of logic limits the scope of this claim concerns their conceptions of validity; in particular, they disagreed on whether the validity of a proof depends on the meanings of the constituent statements. For Frege, content is a logical notion; for Hilbert, thinking contentually about mathematics has a practical value, but the meaning of mathematical terms does not play a role in rigorous proofs. Despite these deep and irreconcilable differences, both Frege and Hilbert viewed the notion of rigorous proof as a guarantee of correctness in mathematics, rather than the source of mathematical justification. In this sense, their conception of proof and its role in proper mathematical method is very similar in spirit, and constitutes one of the most substantial advances in modern logic and mathematics. In concluding, I will suggest that this notion implicitly underlies many mathematicians' ideal notion of proof. While in the near totality of cases proofs in mathematical practice do not live up to these standards, the way of adjudicating controversies in mathematics involves gradually increasing degrees of rigour, mostly conceived as an activity

of filling the gaps in published or informally presented mathematical proofs.

Keywords: mathematical method, mathematical proof, Frege, Hilbert

Ken Archer (Microsoft and Linkoping University). ***From Perception to Probability: A Phenomenological Account of the Origins and Limits of Mathematical Probability.***

Abstract. This paper offers a philosophical reconstruction of mathematical probability as a formalization of pre-mathematical structures of perception and anticipation. Drawing on Husserlian phenomenology and the history of mathematical practice—from Bernoulli and Gauss to modern statistics and artificial intelligence—I argue that probability emerges from the horizonal, anticipatory character of perceptual experience, rather than from frequency counts or subjective belief alone. This account provides a new foundation for interpreting the meaning, success, and limits of probabilistic modeling in mathematics and its applications.

Standard interpretations of probability—frequentist, Bayesian, and symmetry-based—rely on reified constructs (long-run frequencies, degrees of belief, or equipossible outcomes) that presuppose a model structure rather than explain it. In contrast, I propose that probability originates in the open structure of perceptual experience itself. Perception is not a grasp of fully determinate objects, but an active synthesis of incomplete perspectives that are always surrounded by anticipated further appearances. These anticipations are structured by what Husserl calls horizons and propensities—tendencies for how things may appear. This horizonal structure introduces a pre-conceptual, proto-probabilistic dimension to experience, in which some outcomes are more expected than others.

This account reframes the history of probability as the progressive idealization and mathematization of perceptual anticipation. Classical probability, as developed by Bernoulli, formalizes decision-making under uncertainty by freezing anticipated outcomes into a finite distribution of equipossibilities.

Statistical probability, as developed by Gauss, formalizes perceptual uncertainty in measurement by modeling observational deviations as a distribution centered on the most likely value. These mathematical practices encode not only empirical insights but cognitive structures—specifically, the tension between perceptual openness and categorial stability.

I further argue that this phenomenological foundation sheds light on ongoing practices in probabilistic modeling, particularly in data science and machine learning. Neural networks, for instance, operationalize mathematical distributions (via logits and softmax transformations) that capture tendencies toward categorial outcomes—mirroring the propensities of perceptual anticipation. However, the power of these models stems from idealizing what is, in experience, an open-ended and revisable horizon. As such, reified models can obscure the creative ambiguity that drives scientific discovery and human understanding.

This reinterpretation has three implications for the philosophy of mathematical practice:

It highlights the motivational continuity between informal perceptual life and formal mathematical structures.

It offers a non-reductionist epistemology of probability that links mathematical abstraction to cognitive structures without reducing one to the other.

It clarifies the limits and strengths of probabilistic methods in science: they succeed by formalizing aspects of human anticipation, but can also become dogmatic when their idealizations are mistaken for ontological foundations.

Ultimately, this paper situates mathematical probability within the broader project of understanding how formal mathematics arises from and refers back to the life-world, thereby contributing to a practice-oriented epistemology of mathematics grounded in cognitive and phenomenological realism.

Keywords: mathematical probability, phenomenology, perception, artificial intelligence, mathematical distributions

Mark Balaguer (Cal State LA). *Platonism, Mathematical Relativism, and the Mind-Dependence of Mathematical Truth.*

Abstract. In this paper, I will do the following three things: (i) I will argue for a strong version of the idea that mathematical truth is determined by facts about our minds; (ii) I will argue that a certain sort of cultural (or biological) relativism about mathematics is true; and (iii) I will argue that, perhaps surprisingly, these two theses (i.e., mathematical relativism and mind-dependence about mathematics) are compatible with mathematical platonism—and, indeed, that platonists should endorse these two theses. The argument for this last claim is based on the two-pronged claim that (a) platonists should endorse plenitudinous platonism, and (b) the best versions of plenitudinous platonism entail mathematical relativism and mind-dependence about mathematical truth. And the argument for the first two claims—i.e., for the claims that relativism and mind-dependence are true—is based on the claim that the best versions of platonism and anti-platonism both entail these two results.

Keywords: platonism, plenitude, mathematical relativism, mind-dependence, error theory

Antonio Baraldi (Unibo). *THE VALUE AND IMPORTANCE OF INFERENCE TO THE BEST EXPLANATION IN SCIENTIFIC PRACTICE.*

Abstract. The specific research goal I aim to investigate is the ontological status, specific functioning, and epistemological consequences of inference to the best explanation (IBE) as applied to the development of mathematics and physics. This requires a comparative analysis of how IBE operates within these two disciplines, which, while distinct, are complementary.

Mathematics serves as the language employed by physics to describe the natural world, encompassing both observable and unobservable entities. By conducting a comparative analysis of how IBE is utilized in mathematical and physical practices, I aim to deepen the understanding of how hypothesis generation can influence the development of theoretical frameworks, contributing to the debate on scientific realism and anti-realism. Inference to the best explanation represents a highly complex and pervasive form of reasoning within scientific practice. It reflects humanity's attempt to reduce complex physical and metaphysical phenomena within a theoretical framework. Understanding how this form of logical inference works is crucial in order to grasp how we interpret and thereby constitute our major theories. The current state of research is very active and evolving. For example, in mathematical logic, scholars are exploring whether inference to the best explanation (IBE) can be valid in justifying "controversial" or independent axioms of ZFC, such as: the Axiom of Choice (AC), the Continuum Hypothesis (CH) or its negation and Large Cardinal axioms. In set-theoretic pluralism (e.g., Joel Hamkins), it is argued that there are many possible universes (models of ZFC and beyond). In this context, IBE comes into play as follows: we no longer seek an absolute truth (for example if CH is true), but rather the model that best explains certain mathematical phenomena. In this sense, different theories can be locally valid insofar as they successfully explain certain portions of mathematics. Here too, IBE justifies a preference for some models over others, based on their explanatory power or structural elegance. In physics the situation is similar. One example of the current research status is the one of dark matter. The dark matter hypothesis derives from the standard cosmological model and so from the hypothesis (which is the most commonly accepted one) that the Big Bang is the origin of our universe. These brief examples make clear the epistemological importance of analyzing and understanding inference to the best explanation, as it serves both as the "glue" in our attempts to explain and resolve the complex phenomena around us that we do not yet understand, and as the creative process through which we are able to develop new and innovative theoretical frameworks (such as the ongoing work within set theory). These are just a couple of questions coming to my mind:

1. If the existence of some mathematical entities is uncertain, how can we epistemologically ensure that these entities should represent hypothesized physical entities that are experimentally unobservable?

2. What ontological commitments and implications might IBE lead us to in mathematics compared to physics?

Keywords: Epistemology, Philosophy of mathematics, Philosophy of physics

Manuel Barrantes (California State University Sacramento). ***The two-level approach of Mathematical Explanations of Physical Phenomena.***

Abstract. One of the debates concerning so-called mathematical explanations of physical phenomena (MEPPs) revolves around the question of whether mathematics plays a genuinely explanatory role (e.g., Baker 2005) or a mere representational one (e.g., Leng 2021). In a recent paper, Baker concedes that at one level these explanations use representational models, but this does not disqualify them from being MEPPs. He argues that, once these models are implemented, a purely mathematical reasoning occurs at a higher level, which is what really explains the explanandum, thus rendering the whole explanation a genuine MEPP (Baker 2021, 15) (Huneman 2018 holds a similar view).

Baker introduced this distinction with the bipedal gait explanation, which consists in explaining the 3:1 ratio between the energy consumed at the stance phase and the swing phase of bipedal animal walking. For Baker, straightforward representational models capture the energy costs of the different phases (level 1). But the ratio itself is calculated by a minimum value theorem that not only proves it, but also explains it (Baker 2021, 7), which is what makes this case a MEPP (Baker 2021, 16).

I think, however, that this explanation can also be understood from a representationalist perspective. The minimum value theorem is not used as a black box that provides results detached from physical reality. Rather, it is understood as tracking down the relevant physical relationships that explain the invariant ratio. The higher-level reasoning occurs precisely because the level 1 use of mathematics successfully captures the relevant stages of walking. But there is no guarantee that the higher-level reasoning will continue to be successful in the same way, because there is always the possibility that it will bring extraneous results that do not necessarily make sense for the physical system. This is acknowledged by the author of the study cited by Baker, who pointed out that his model also predicted a third optimal gait that, he thinks, may be a mere artifact (Srinivasan 2006, 108). Since the higher-level mathematical results are explicitly interpreted in empirical terms (Srinivasan 2006, 106), this supports the view that they play a representational role as well.

Against this, it may be pointed out that Srinivasan's study includes a mathematical explanation of the minimization theorem (Baker 2021, 11-12; Srinivasan 2006, 104), and that this would show that the overall scientific explanation depends on a mathematical explanation, rendering this case a genuine MEPP. However, this objection is incompatible with Baker's stance in a related debate, namely, the view that MEPPs only need to cite, but not explain, the mathematical theorems they use (Baker 2012). The objection is compatible with an alternative view though, that Baker rejects, which holds that MEPPs must include purely mathematical explanations (Steiner 1978; Colyvan 2018). However, as Barrantes (2020) has pointed out, and scientific practice seem to suggest (Baker 2012), the merits of these derivations (e.g., whether they qualify as purely mathematical explanations) are unimportant for empirical purposes.

References

Baker, A. (2005), 'Are there Genuine Mathematical Explanations of Physical Phenomena?', Mind, 114: 223–238.

Baker, A. (2012), 'Science-Driven Mathematical Explanation', Mind 121 (482): 243-267.

Baker, A. (2021), ‘Bipedal Gait Costs: a new case study of mathematical explanation in science’, European Journal for Philosophy of Science, 11/3: 1-22.

Barrantes, M. (2020), ‘Explanatory Information in Mathematical Explanations of Physical Phenomena’, Australasian Journal of Philosophy, 98 (3), 590-603.

Colyvan, M. (2018), ‘The Ins and Outs of Mathematical Explanation’, The Mathematical Intelligencer, 40, 26–29.

Huneman, P. (2018), ‘Outlines of a theory of structural explanations’, Philosophical Studies, Volume 175: 665–702.

Leng, M. (2021), ‘Models, structures, and the explanatory role of mathematics in empirical science’, Synthese 199 (3-4): 10415-10440.

Srinivasan, M. (2006), Why Walk and Run: Energetic Costs and Energetic Optimality in Simple Mechanics-based Models of a Bipedal Animal. PhD Dissertation. Cornell University, 370 pages.

Steiner, M. (1978), “Mathematics, Explanation, and Scientific Knowledge”, Noûs, 12: 17–28.

Keywords: Mathematical Explanations, Bipedal gait cost, Representation vs explanation

Rachel Boddy (IUSS Pavia) and Robert May (University of California, Davis). *Logic vs. Logicism.*

Abstract. “The introduction of value-ranges of functions is an essential step forward”, Frege says in describing the innovations introduced in the logical system presented in Grundgesetze. The reason, Frege tells us, is because “I define cardinal numbers themselves as extensions of concepts, and extensions of concepts are value-ranges, according to my specifications.”In this talk, we discuss how Frege operationalizes value-ranges to logically characterize the core mathematical notion of logicism, of a concept having a number, by defining cardinal numbers as value-ranges of concepts.

Central to Frege’s analysis is that concepts are logically characterized as functions, and that value-ranges are objectual correlates of functions. This reification allows Frege to address a problem in his logical language, its inability to express the conceptual identities required for the scientific content of the logicist program. Frege addresses this lacuna by having value-ranges represent functions, so that these identities can be indirectly expressed by value-range identities. This representational role relies on intensional aspects of concepts, and leads Frege to deny that value-ranges terms refer to classes, creating a tension with the truth-conditions of these identities, specified extensionally, by Basic Law V, as identity of the graphs of the functions corresponding to the value-ranges. But Basic Law V is itself fatal; conjoined with how Frege represents the predicativity of concepts with value-ranges, it leads directly to Russell’s Paradox. And with this, Frege’s answer to “the fundamental problem of arithmetic: how are we to apprehend logical objects” is “shattered”.

Keywords: Frege, Logic, Logicism, Concept, Function, Value-range

Moritz Bodner (University of Vienna). *The Significance of Normal Form Results.*

Abstract. I propose to examine the role normal form results play in mathematical practice. Given the ubiquity of such results reflecting on them is of obvious interest to the philosophy of mathematical practice.

What is more, besides there purely mathematical significance – as results about the structure and (consequently) possible classification of mathematical objects of a certain kind – normal form results have distinctive philosophical and methodological significance.

To highlight them, I will mine the emergency of the Decision Problem for case studies:

Heinrich Behmann, in his 1922 article introducing what he called the "Entscheidungsproblem", cites Hilbert's use of (disjunctive and conjunctive) normal form results for sentential logic as a prototype of a decision procedure for deductive systems. Apart from serveing as inspiration for further work, normal form results seem to have functioned in this case, as I believe they often do, as what I shall call "epistemic catalysts". They provide knowledge about a certain domain, which is rich and striking in a way that gives rise to further knowledge about the domain: By showing every object in a domain (in this case: every formula of propositional logic) to be equal (by the lights of that domain, i.e. truth-functionally equivalent) to an object exhibiting a distinctive (syntactic) structure, normal form results provide knowledge of the (philosophically speaking) "nature" or "essence" of these objects which is more specific than what is known about the objects merely on account of how they are generated (by construction or axiomatic characterisation); this additional information about structural features of the objects allows one to infer further facts about the objects (in this case: their truth or falsity) and also (since every such object has such a normal form) about the domain as a whole (in this case: the decidability of sentential logic). (A similar role is played by the prenex normal form result in Gödel's proof for the completeness for first-order logic.)

This explanation brings to light a second notable feature: Normal form results have an intensional flavour. The richness of the knowledge about the domain they provide is due to the specific representations they provide, picking out the objects in a given domain by means of a term of a "normal" or "canonical" form; this value of a normal form result is not preserved when another representation is used, even if the alternative mode of representation is extensionally equivalent.

I will illustrate this intensional character by recalling Gentzen's normal form result for deductions (his "Hauptsatz"), which enabled Gentzen to draw conclusions about consistency and decidability (of logic) from finding that all provable formulas can be represented in a certain way (as last lines of proofs using the available means of proof in ways more restricted than than what the underling definition of proof allows for in principle).

Finally, normal form results also play a crucial methodological role on a higher level in facilitating the drawing of connections between different disciplines. This I will illustrate by way of Kleene's normal form result for recursive functions as an exmaple.

Keywords: normal form results, history of logic, intensionality

Nicolò Cambiaso (University of Statale di Milano- School for Advanced Studies IUSS Pavia) and Tommaso Peripoli (University of Statale di Milano- School for Advanced Studies IUSS Pavia).

Changing Mathematical Concepts: Two Modes of Revision.

Abstract. Recent works on Lakatos' conception of mathematical concepts have defended the idea that mathematical concepts can undergo revision (Schlimm 2012, Tanswell 2018, Shapiro & Craige 2021). We aim at contributing to this view by making two novel claims. First, we identify two distinct modes of revision in mathematics – an analytical and intra-theoretical mode; second, we argue that mathematical concepts differ in how they respond to revision.

Analytical revisions aim to uncover the true meaning of mathematical terms and are often close-ly connected to a metaphysically loaded understanding of mathematics. In contrast, intra-theoretical revisions are concerned finding the best definition of a concept for the theory at hand, without re-course to metaphysical commitment. We illustrate this distinction with two case studies: the XIX de-bate over Euler's formula for polyhedra and the historical development of differentials.

The debate over Euler's formula, recounted in Lakatos' Proofs and Refutations, provides clear examples of analytical revision. In response to counterexamples that violated Euler's formula, mathe-maticians proposed revised definitions of "polyhedron" to preserve what they saw as its true nature. By contrast, the historical treatment of differentials—introduced by Leibniz and revised by figures like Lazare Carnot—illustrates intra-theoretical revision. Though inconsistent, mathematicians continued to use differentials as useful tools. Thus, when revisions were attempted, mathematicians focused on im-proving internal coherence rather than uncovering the true nature of differentials.

Drawing from these and other examples, we show how mathematical notions respond differ-ently to revision. Notions like differentials are not usually subject to analytical revision. By contrast, concepts such as polyhedral, geometrical figures, natural numbers have been analytically revised or both analytically and intra-theoretically revised. The distinction between modes of revision offers a fruitful framework for investigating the development of mathematical concepts both within and across different theories.

It is valuable to understand how mathematics has practically revised its own concepts to create a useful framework for exploring the development of mathematical ideas. We aim to demonstrate, through two historical examples, that there are at least two distinct types of revisions.

References

• Lakatos, Imre. Proofs and refutations: The logic of mathematical discovery. Cambridge uni-versity press, 2015.

• Schlimm, Dirk. "Mathematical concepts and investigative practice." Scientific concepts and in-vestigative practice 3 (2012).

• Shapiro, Stewart, and Craige Roberts. "Open texture and mathematics." (2021): 173-191.

• Tanswell, Fenner Stanley. "Conceptual engineering for mathematical concepts." Inquiry 61.8 (2018): 881-913.

Keywords: Conceptual Change, Lakatos, Infinitesimals, Lazare Carnot

Jean-Paul Cauvin (University of Pennsylvania). *Style as Mathematical Practice in the Philosophy of Gilles Gaston-Granger.*

Abstract. I argue that the notion of style in the work of Gilles Gaston-Granger (1920-2016) can be productively interpreted as a theory of mathematical practice.

The paper is primarily historical, and reconstructs three different articulations of the epistemological significance of style in Granger's work. The first articulation of style encompasses what Granger describes as the schematization of relations of form and content in the history of science. Granger develops this notion of style in Pensée formelle et sciences de l'homme (1st ed. 1960, 2nd ed. 1967, Tr. Eng. 1983). The second articulation of style is perhaps the most well known, at least in the Francophone historiography, in which style becomes both a descriptive and an interpretive device again indexed to the relation of form and content, but this time in the service of a generalized project of comparative epistemology as developed in the Essai d'une philosophie du style (1968). Finally, in what might be described as Granger's "metaphilosophy", style is again reconceptualized as a "transcendental ergology" or "transcendental theory of the work of the symbol" in Pour la connaissance philosophique (1988), Formes, opérations, objets (1994), and Sciences et réalité (2001).I argue that each articulation of style expands the remit of what Granger seems to understand by practice, especially with regard to conceptual transformations in the history of mathematics. I suggest that contemporary theorists of mathematical practice, and indeed the broader tradition of anglophone philosophy of mathematics, might profitably engage with Granger's philosophy of style and comparative epistemology.

Keywords: Style, Gilles Gaston-Granger, Aesthetics of Mathematical Novelty, Semiotics of Mathematical Expression, Historical A Priori, Mathematical Style and Philosophical Style, Comparative Epistemology of Mathematics

William D'Alessandro (William & Mary). *Using Large Language Models to Study Explanation in Mathematics.*

Abstract. The philosophy of mathematical practice (PMP) looks to evidence from working mathematics to help settle philosophical questions. One prominent program under the PMP banner is the study of explanation in mathematics, which aims to understand what sorts of proofs mathematicians consider explanatory and what role the pursuit of explanation plays in mathematical practice. Most research on this topic has taken the form of small-scale case studies. In an effort to address worries about cherry-picked examples and file-drawer problems in PMP, a handful of authors have recently turned to corpus analysis methods. But this research has itself been criticized for attempting to draw philosophical conclusions from crude counts of ambiguous terms like 'explains' and its cognates.

What's needed to propel such methods to greater success, it seems, is a language-processing tool capable of taking in large amounts of text and reasoning insightfully about the meanings it contains to a high degree of accuracy. The current generation of frontier language models is already adept at sophisticated text analysis. The main bottleneck, now on the verge of being overcome, has been the capability for researchers to feed such systems big-data-sized text corpora. This talk reports the results from such a corpus study facilitated by Google's Gemini 2.5 Pro, a model whose million-token context window allows for the analysis of hundreds of pages of text per query.

The experiment, based on a sample of 5000 mathematics papers from arXiv.org, aimed to gain insight on questions like the following: How often and in what terms do mathematicians make claims about explanation in the relevant sense? What philosophical theory of explanation are these uses most consistent with? Do mathematicians' explanatory practices vary in any noticeable way by subject matter, problem type or otherwise? As the first PMP study of its kind (to the author's knowledge), it also seeks to begin a conversation about LLM methods as research tools in practice-oriented philosophy and to evaluate the strengths and weaknesses of current models for this style of work.

Keywords: Artificial intelligence, Large language models, Mathematical explanation, Corpus analysis, Big data, Methodology of PMP

Silvia De Toffoli (University School for Advanced Studies IUSS Pavia) and Elijah Chudnoff (University of Miami). *Articulate Intuition.*

Abstract. A common thought about intuition and inference is that they have contrasting epistemologies: intuition, like perception, purports to immediately justify belief, while inference is a way of basing a belief on supporting considerations. This alignment of intuition with perception in opposition to inference encourages the idea that intuitions cannot be fruitfully shaped by rational reflection, and this in turn fuels various skeptical challenges to reliance on intuition in mathematics (and philosophy). In this talk, we argue that some intuitions, which we call articulate intuitions, share key epistemic benefits associated with inference while continuing to provide immediate justification like perception. In particular, we show that intuitions do have a proper place in mathematical practice and are not in tension with rigorous proof.

We show that the opposition between intuition and inference is a false opposition. We agree that intuition, unlike inference, provides immediate justification. However, we explore other ways in which intuition is, at least in some cases, epistemically unlike perception and more like inference. To a first approximation, our thesis is that some intuitions share key epistemic benefits with some inferences because each derives from an appropriate relation to an argument. The inferences are made by following an argument. We'll call these explicit inferences. The intuitions are enabled by thinking through an argument. We'll call these articulate intuitions. In order to characterize articulate intuitions, we take into consideration a case study from knot theory.

There are potential anticipations of our line of thought in Descartes and Felix Klein: they both saw a possible convergence of intuition and inference, albeit from opposite directions. Descartes argued that the best inferences ultimately become intuitions, while Klein argued that the way to improve intuitions is to make them more like inferences.

Keywords: intuition, inference, mathematical intuition, rigor, knot theory

Eamon Duede (Purdue University). *Collective Creativity in Mixed-Agent Science.*

Abstract. Scientific research often exhibits features of collective agency. Teams of scientists carrying out large-scale experiments are organized in such a way as to distribute cognitive labor across distinct tasks. However, quite often the agents themselves are distributed, as is the case with many large-scale experiments where tasks are carried out by teams peppered across great distances.

In many such cases, individual scientists who make up the group are not in a position to directly evaluate the epistemic status and justificatory credentials of the work of their colleagues, despite their own work depending on these credentials for justification. As a result, within particular investigations and throughout science, epistemic trust is ubiquitous. Fortunately, a maturing literature has gone some way toward excavating the epistemological foundations for this trust.

Through engagement with the epistemology of computation, these foundations have only just begun to clarify and support our understanding of how groups can heavily but justifiably rely on computational processes such as calculation, simulation, and AI. However, this literature is characterized principally by its focus on and contribution to epistemological concerns specifically regarding justification. Meanwhile, philosophers of science have paid very little attention to distributed processes of AI-infused discovery. In this talk, I focus on one aspect of discovery that stands to be clarified in light of generative AI ---collective creativity.

Philosophers of mathematics have long been interested in creativity in the practice of mathematics. Moreover, since the proof of the four-color theorem, which relies crucially on computer assistance, philosophers have debated (and at times thought to revise) the nature of proof itself. However, while the use to which computation is put in the proof of, for instance, the four-color theorem is properly creative, the role played by computation in that proof certainly is not.

However, several recent mathematical advances have leveraged artificial intelligence in such a way that the AIs themselves carry out much of the creative burden. This talk centers around two cases in which what would count as significant creative cognitive labor in a human agent is carried out by an AI working alongside humans. The first leverages AI to guide mathematical intuition in low dimensional topology toward promising avenues to prove conjectures relating geometric and algebraic properties of knots. The second leverages generative models in an evolutionary procedure to make breakthroughs in the cap set problem in extremal combinatorics, leading to new discoveries that surpass previously known results.

Given that these tasks, if carried out by a human, would be considered creative, I aim to make sense of whether and to what extent these computations should, likewise, be considered creative. Moreover, in both cases, the model's output is passed to a human mathematician, not to be evaluated, but to be built upon in a way analogous to the collaborative act of collective discovery. I aim to make sense of these moments in the process as well, and to evaluate the extent to which reflection upon them changes our conception of the creative interplay that unfolds between collaborating scientists in the otherwise routine execution of collective discovery.

Keywords: Collective Mathematical Creativity, Epistemic Trust, AI-Infused Mathematical Practice

Kenny Easwaran (UC Irvine). *Several AI futures for mathematical proof.*

Abstract. Since the early 20th century, there has been an uneasy relationship between mathematical proofs, as published, and formal proofs, as defined in various systems like first-order ZFC or various forms of type theory. Some philosophers have argued that published mathematical proofs derive their status from their association with formal proofs, but Easwaran (2008, 2015) argued that they do not.

Instead, the standards of the mathematical community seem to be that published proofs should be "transferable" (they should enable members of the target audience to generate autonomous knowledge of the conclusion, without essentially depending on trusting the author), and "convertible" (spelled out at a level of detail that is sufficient to convert any possible rebutting defeaters (potential counterexamples) into undercutting defeaters (demonstrations that some particular step in the argument fails)). This is the standard even though most readers do trust the author for at least some results, rather than forming their own autonomous knowledge, and even though most published proofs never have any rebutting defeaters.

He argued that these standards distinguish math from empirical sciences, where the standards are not so demanding. In empirical sciences, readers expect to have to trust the author - though they ask that there be some description of the methods. Probabilistic statistical tests are accepted, even though statistical tests are not convertible. He argued that mathematicians insist on these stronger standards, while not insisting on the production of a formal proof, in part because of some contingent facts about the level of difficulty of the work. Lower standards would result in too much publication, while higher standards would result in too little.

The rise of artificial intelligence threatens to change this situation in several ways. In the past few decades, proof assistants like Coq, Isabelle, and Lean have made formal proof more accessible than it was through most of the 20th century, though not enough so to make it practically accessible for most mathematical results. Meanwhile, generative AI language models have made it easy to generate text that looks like standard mathematical proofs (which may or may not rise to the level of what de Toffoli calls "simil-proofs"), but haven't made it any easier to check whether they meet the standards for publication. This threatens to overwhelm the refereeing process at journals. Depending on future developments, there are several directions the mathematical community could go.

If similar AI tools enable the automated conversion of ordinary proofs into formal proofs ("autoformalization"), one natural response would be to insist on including formal proofs together with ordinary proofs in publication, raising epistemic standards. If autoformalization remains out of reach, a suggestion by Emily Riehl is that formalization might be required for proofs constructed by language models (though it is unclear how such a requirement could be enforced). But I argue that another rational possibility may turn out to be the embrace of the lower epistemic standards of the empirical sciences.

Keywords: mathematical proof, AI, publication norms, formalization

Don Fallis (Northeastern University). *What's So Special about Deductive Proof?.*

Abstract. Mathematics is famously a *deductive* science (at least as far back as Euclid 300 BCE). That is, in order to establish that a mathematical claim is true, mathematicians *prove* it. And this is not just a matter of stylistic preference. We tend to think that deductive proof is better at securing mathematical knowledge than inductive evidence. The traditional view is that, although mathematicians sometimes make mistakes even when they restrict themselves to deductive proof, deductive proof is *more reliable* than inductive evidence.

However, in the last fifty years or so, mathematicians have developed many Monte Carlo methods for checking mathematical claims (see Rajeev and Raghavan 1996). For example, there are “probabilistic proofs” that can establish with arbitrarily high probability that a number is prime (e.g., Rabin 1980). As a result of these developments, some mathematicians (e.g., Zeilberger 1993, Hersh 1997, 56-59, Borwein 2008) claim that mathematicians should use probabilistic proofs to establish that mathematical claims are true when such extremely reliable inductive evidence is available. Moreover, a few philosophers (e.g., Fallis 1997, Corfield 2003, 109-10, Womach and Farach 2003, Paseau 2015, Brown 2020) have argued that there is no epistemic (i.e., knowledge-related) value of deductive proof that inductive evidence always lacks.

In response, several philosophers (e.g., Avigad 2008, 307, Easwaran 2009, Smith 2016, 48-50, Berry 2019, Hamami 2022, Hamami 2023, Lange 2024) have subsequently tried to identify the distinctive epistemic value of deductive proof. These proposals can definitely help us to understand why deductive proof is as reliable as it is. But as I will argue, since the identified properties of deductive proof are *only* valuable as a means to reliability, these proposals are ultimately unsatisfying. They do not provide us with any reason not to use inductive evidence when it is just as reliable as deductive proof in establishing that a mathematical claim is true. In other words, we have yet to identify the distinctive epistemic value of deductive proof.

Keywords: Deductive Proof, Mathematical Knowledge, Probabilistic Proof

Lyu Fu (IHPST). *The Inferential Account and the Confidence Objection.*

Abstract. How can we understand the epistemic efficacy of mathematics in empirical science? The Inferential Account of applied mathematics maintains that the fundamental role of applied mathematics is inferential, and that it is precisely by virtue of this inferential role that mathematical methods contribute to prediction, explanation, and unification within scientific practice. Nonetheless, the notion of "inference" within this framework remains opaque. According to the Inferential Account, prediction, as an exemple, proceeds via three successive operations: an empirical system (or its description) is embedded within a mathematical structure ("immersion"); formal derivations are carried out therein ("derivation"); and the results of these derivations are re-interpreted as assertions about the empirical system ("interpretation"). Given the pragmatic nature of the immersion and interpretation steps, this tripartite procedure does not itself instantiate a logical inference in the strict sense—that is, a object possessing an explicit premise–conclusion form. This stands in contrast to the standard conception of prediction, whereby a claim qualifies as a prediction of a theory only if the former follows deductively from the later.

This tension can be recast as a "confidence objection": if the Inferential Account is a good account for mathematics' contributions to unification, prediction, and explanation, then the link between a prediction, as example, and its generating theory cannot be a genuine logical inference. Consequently, the account seems unable to explain for what reasons scientists trust their predictions.

To evaluate this concern, I will analyze the Inferential Account's proponents' treatment of the confidence objection, which centers on Dirac's prediction of the positron and interrogates whether Dirac possessed adequate epistemic grounds to trust his own prediction.I argue that (1) despite potential disputes over historical minutiae, the claim that Dirac lacked sufficient reason to trust his prediction remains defensible; yet (2) the assertion that the relevant interpretation is underdetermined by the formalism encounters substantive counter-evidence.

By contrasting the Inferential Account with Steiner's Analogy Account and Pincock's Mapping Account, I demonstrate that the Inferential Account struggles to articulate the inferential structure underpinning scientific practices of prediction, explanation, and unification. Its schematic bifurcation into purely "mathematical" and "empirical" domains leaves no home for the mixed statements—statements that the Mapping Account, in contrast, explicitly accommodates.

In response to the confidence objection, I propose that the Inferential Account can offer two replies. First, the Inferential Account remains compatible with the view that certain theory-to-claim relations are inferential. Second, more fundamentally, the account is not designed to expound those inferential relations in full logical detail; rather, it aims to elucidate the role that mathematical inference plays within scientific reasoning. Moreover, once this role is clarified, it becomes clear that the ostensible inferential form of these relations is merely superficial, concealing a wealth of pragmatic considerations.

Keywords: Philosophy of mathematical practice, Application of mathematics, Inferential Account

Lorenzo Gandolfi (IUSS Pavia). *Choosing what to prove and how.*

Abstract. In this talk, I will present a model of how mathematicians choose which statements to prove and which strategy to adopt to do so. I will then apply the model to the case where the method of reductio is used and show that the model's predictions align well with evidence from literature in mathematics education.

Unlike ideal agents, humans have to carefully manage their cognitive and time resources, since they have limited availability of both. Thus, when engaging in a mathematical practice, in which often success is linked to proving certain statements, they try to determine which statements are worth a try and which proof strategy is contextually the best suited. According to my model, an agent S, considering whether to try to prove that p, will conduct an initial evaluation of the evidence she considers relevant to determine her preliminary credence for p. S will try to prove that p just in case her preliminary credence for p turns out to be above a certain threshold. The importance of mathematicians' beliefs in still unproved statements has recently been emphasized by Gowers (2023). Of course other factors can play a role as well in whether or not to actually give it a try. I postulate that the initial examination of the evidence will also provide S with a ranking of the proof strategies she knows. She begins trying to prove the candidate theorem with the first-ranked strategy and moves on to the others if, after various attempts, she has not achieved anything.

Let us consider a case in which a mathematician ends up trying to prove that p by reductio. As the strategy prescribes, she will then assume not-p, typically with some uncontroversial additional premises, and try to revise her web of beliefs (WoB) accordingly. S already has a relatively high credence for p, in this context. So not-p here counts as what Rescher (1961) would have called a belief-contravening supposition (BCS). A BCS is an assumption that pulls in the opposite direction to pre-existing beliefs and, according to Rescher, does not imply any specific set of changes. S then knows she has to manipulate her WoB but no automatic method can tell her exactly how, leaving her in a stalemate.

So, the model predicts that S will experience a sort of intellectual paralysis. While this does not seem to be the case for professional mathematicians, who are well trained and therefore able to easily overcome these cognitive difficulties, students encounter significant issues when working with reductio proofs, as pointed out by educators.

The prediction of my model, in fact, align with evidence from mathematics education literature (Leron 1985; Antonini & Mariotti 2008; Turiano & Boero 2019). This provides support for its validity.

References

Antonini, S., & Mariotti, M. A., 2008, ZDM 40(3), 401–412.

Gowers, T., 2023, Ann. Math. Philos. 1(1).

Leron, U., 1985, Educ. Stud. Math. 16(3), 321–325.

Rescher, N., 1961, Philos. Rev. 70(2), 176.

Turiano, F., & Boero, P., 2019, CERME 11, Utrecht, Netherlands, hal-02398527.

Keywords: Proofs, Epistemology, Education

Eduardo Giovannini (Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET)) and Georg Schiemer (Department of Philosophy, University of Vienna). *Formal Content and Equivalence in Mathematical Practice.*

Abstract. Mathematicians often speak informally about the 'content' of specific mathematical theorems or statements. For instance, it is often said that Fermat's Last Theorem has a purely arithmetical content because it only talks about integer solutions to a specific equation, even though its proof has revealed deep connections between algebraic geometry and number theory. Likewise, Euler's polyhedron theorem is often considered to have a strictly geometrical content, as it describes relations among vertices, edges, and faces of a convex polyhedron, despite its deep correlations to topology and graph theory.

The idea of content is thus closely related to the account of the meaning of statements in mathematics and has featured a prominent role in recent debates in the philosophy of mathematical practice, particularly in discussions on the purity of mathematical proofs ([1],[2],[3]).

More precisely, one classical conception of purity is articulated in terms of the idea of the content of mathematical statements: ''in modern mathematics [...] the aim is to preserve the purity of the method, i.e., to prove theorems if possible using means that are suggested by the content of the theorem'' ([4, pp. 315-6]). Two primary approaches to understanding the mathematical content of statements have been then proposed in the recent literature. The first grants a privilege to the informal or ''intuitive'' understanding of mathematical languages, and amounts to what someone with a casual acquaintance with mathematics would understand by some statement. In contrast, the second approach presupposes a ''formal'' understanding of mathematical languages and identifies content with the meaning of mathematical statements within the context of abstract axiomatic theories formulated in logical languages. These are two very different accounts of the notion of the content of a mathematical statement.

Recent investigations into the problem of the purity of proofs in mathematical practice have primarily focused on the intuitive conception of content, leaving the formal notion relatively imprecise. This talk has a twofold aim. First, we will provide a more systematic conceptual analysis of the idea of formal mathematical content by providing a precise logical explanation. More specifically, using an illustrative example stemming from elementary geometry, namely the famous Desargues' theorem, we will identify three constraints for a formal account of mathematical content: theory relativity (I), independence (II), and non-reducibility (III). Based on this, we will offer two explications of the formal content of a statement relative to an axiomatic theory, an inferentialist and a model-theoretic one. Second, drawing on fundamental geometrical and algebraic results in the modern theories of affine and projective planes, we will establish two notions of equivalence of formal mathematical content.

We will then argue that these criteria of sameness of formal content are instructive for a better understanding of claims about the equivalence of statements across different theoretical contexts in mathematical practice.

Keywords: formal content, mathematical practice, axiomatic geometry, theoretical equivalence, meaning

Matt Haber (University of Utah). ***Positively Misleading Errors.***

Abstract. In 1978 the biologist Joseph Felsenstein published the paper, "Cases In Which Parsimony Or Compatibility Methods Will Be Positively Misleading" in the journal *Systematic Zoology*. In it he demonstrates that popular methods for reconstructing evolutionary histories ('phylogenies' or 'evolutionary trees') will, under certain specified conditions, systematically yet erroneously lump together taxa as closely related when they are, instead, separated by long evolutionary branches. This erroneous behavior has been dubbed 'long-branch attraction' and identified as a distinctively challenging kind of statistical inconsistency, which Felsenstein called a 'positively misleading error'.

Statistical methods have the property of consistency when they converge on the correct outcome as more data accumulate; statistical inconsistency is the failure to converge on that outcome. Felsenstein observed that inconsistency may be expressed as convergence on an incorrect outcome (as opposed to just failing to converge at all). The positive support attributed to the erroneous outcome comes at the expense of the correct one, mimicking statistical consistency, and, left undiagnosed, can mislead researchers about the systems they are studying. Hence, these errors are positively misleading.

My goal is to provide a more general account of positively misleading errors (PMEs), arguing that they are a distinctive and important category of statistical or probabilistic reasoning.

Outside of a few fields of biology, researchers are generally less familiar with PMEs than other errors of statistical reasoning, e.g., type I errors or base-rate fallacies. Drawing attention to PMEs will help us gain a better understanding of them and provide resources to researchers to more effectively identify and dislodge these errors, much as we have gained a better understanding of good scientific reasoning from studying other errors of statistical reasoning (e.g., Mayo 1996).

Here, I introduce PMEs with an idealized, cartoon case. This is intended as an entry point into phylogenetics---the field of biology that includes reconstructing evolutionary history---setting the stage for a presentation of the case from which Felsenstein first identified and described positively misleading errors. Following a brief discussion of why PMEs are distinct from more familiar errors of statistical reasoning, I explore some of the philosophically interesting features of PMEs. That includes how PMEs suggest adopting a minimal methodological pluralism and the way PMEs can generate 'epistemic traps for researchers.

PMEs are unlikely to be limited to phylogenetics. To demonstrate this I consider a candidate case from clinical medicine and submit other possible cases. This suggests that PMEs may be more widespread than appreciated and highlights what is at stake in understanding them. There are important and pressing consequences of this error of statistical and probabilistic reasoning and we ought to seek ways to improve our ability to identify, diagnose, and dislodge PMEs. A good starting point is naming the problem and identifying exemplar cases we may study and learn from.

Keywords: Probabilistic Reasoning, Statistical Reasoning, Positively Misleading Errors, Epistemic Traps, Pattern Recognition, Scientific Methodology, Phylogenetic Inference, Clinical Treatment Protocols

Joshua Hunt (Syracuse University). ***Making properties Manifest in Mathematics.***

Abstract. A long-standing issue in the philosophy of mathematical practice involves making sense of the value of re-proving theorems already known to be true (Dawson 2015). Mathematicians devote tremendous resources to constructing alternative proofs of previously proven theorems. Clearly, the intellectual value provided by new proofs of old theorems goes beyond gains in evidence or justification. As part of this broader issue, I will focus on how certain conceptual or notational choices can succeed at making properties manifest that were previously hidden or obscured. I will argue that in at least some cases, part of the value of re-proving comes from making properties more manifest. My talk will draw on recent work by Morris (2020) on well-motivated proofs, Morris and Hamami (2021) on viewing proofs as plans, and Hunt (forthcoming) on the value of reformulating problem-solving procedures.

When mathematicians call a property ‘manifest’, they roughly mean that it is easy or simple to infer that the property obtains. In some cases, an apt choice of representational means allows one to ‘read of □’ the property from a written expression. Yet, what it takes for an inference to be ‘easy’ or ‘simple’ is subjective: what is easy for one person to infer might be difficult for another, based on differences in their cognitive capacities or skills. Fortunately, from this subjective starting point, we can extract a notion of ‘manifestness’ that is at least intersubjective. I will propose and defend a more precise characterization of what it means to make a property manifest, illustrating my account with examples from mathematics and logic (including graph theory, Euclidean geometry, and truth tables vs. truth trees).

Lying behind mathematicians' intuitions about easy or simple inferences are norms governing when an inference is warranted. I propose that a formulation makes a property manifest when an agent who understands that formulation is warranted to infer that the property obtains. In other words, the agent's evidence warrants or licenses this inference. Depending on one's preferred account of epistemic warrant, this makes 'manifestness' at least intersubjective—if not entirely objective. Additionally, by focusing on epistemic warrant, my account avoids appeals to differences in how joint-carving, natural, or fundamental different notations or concepts are. In this way, my proposal is less metaphysically-committed than related proposals regarding perspicuous representations (Møller-Nielsen 2017; North 2021).

Since formulations can make properties more or less manifest, my account must allow for gradation. Intuitively, depending on the formulation (e.g. concepts or notation) being used, an agent can be more or less 'close' to having a warranted inference. One way to quantify this notion of closeness relies on the number of epistemically possible solutions at a given stage of problem-solving or proof construction. A solution counts as epistemically possible provided it has not yet been ruled out by the agent's evidence. Other things equal, an agent is closer to the solution provided fewer non-actual answers are epistemically possible, i.e. provided their evidence has ruled out more non-actual putative solutions. At each stage of calculation or step in a proof, an agent applies an inference rule which changes their evidence (or at least their proximal or salient evidence), restricting which answers are epistemically possible (relative to their proximal evidence). This suggests the following gradated account of manifestness: compared to a given formulation or proof, another makes a property more manifest provided it rules out more epistemically possible properties from the given contrast class.

References

Dawson John W., J. (2015). Why Prove it Again? Alternative Proofs in Mathematical Practice. Cham:Birkhäuser.

Hamami, Y. and R. L. Morris (2021). "Plans and Planning in Mathematical Proofs". In: The Review of Symbolic Logic 14.4, pp. 1030–1065.

Hunt, J. (forthcoming). "On the Value of Reformulating". In: Journal of Philosophy. Møller-Nielsen, T. (2017). "Invariance, Interpretation, and Motivation". In: Philosophy of Science 84, pp. 1253–1264.

Morris, R. L. (2020). "Motivated Proofs: What they are, Why they Matter and How to Write them". In: The Review of Symbolic Logic 13.1, pp. 23–46.

North, J. (2021). Physics, Structure, and Reality. Oxford: Oxford University Press.

Keywords: reformulation, proofs, understanding, conceptual change, manifest properties

Mahmoud Jalloh (California Institute of Technology). *Descartes' Dimensions and the Euclidean Taboo.*

Abstract. Recent work has reevaluated the results and method of Descartes' La Geometrie in light of his systematic philosophy as first promulgated in his Regulae (Liu 2017; Moon 2023). These contributions touch upon but do not totally resolve a further ambiguity in Descartes' development of analytic geometry. My concern is his handling of what I'll call the "Euclidean taboo". This taboo is rooted in the third definition of book V of the Elements, which restricts the ratio relation to holding only between magnitudes of the same kind. Understanding "same kind" to mean (at least) "same dimension", this taboo is precisely the issue that prevented the ancients from applying algebra (or "arithmetic") to geometry, according to Descartes. In some places Descartes seems simply to reject the taboo altogether and allows his equations to add and equate quantities of different dimensions—these quantities are to be interpreted as simple lines, regardless of their degree. This would be a straightforward rejection what we now call the principle of dimensional homogeneity. However, Descartes has an aside in book I of La Geometrie which belies his apparent rejection of the principle: there he makes the claim that his approach can satisfy the principle of dimensional homogeneity, thereby obeying the Euclidean taboo.

Descartes obedience or disavowal of the principle of dimensional homogeneity seems to turn on whether or not the unit in question is determined by the problem or freely chosen. The significance of this condition is to be interrogated.

My aim is to explain what Descartes competing arguments regarding dimensional homogeneity are, anachronistically using the tools of contemporary dimensional analysis. On the one hand, the suitability of this analysis will provide an argument regarding an ongoing question regarding the foundations of dimensional analysis-Descartes approach will be contrasted with more recent approaches. On the other hand, this exploration on makes more perplexing Descartes apparently geometry-first conception of his unification of geometry and algebra.

Keywords: Descartes, dimensional analysis, analytic geometry

Bendix Kemmann (Department of Philosophy, Stanford University). ***Instruments of the mind: A theoretical and computational study of numerical notation systems.***

Abstract. Humans can estimate approximate quantities independently of language or symbolic number systems. However, to perform precise calculations we rely on notations for numbers. Symbolic number systems are 'instruments of the mind': they allow us to form thoughts and make inferences that go beyond what we can do with our bare brains, just like physical instruments extend what we can perceive and do. In this talk, I show that number notations facilitate computation in virtue of syntactic relations among numerals that both preserve and reflect arithmetic relations among numbers. In other words, number notations do not merely represent structure — they instantiate it via partial embeddings. This allows for the manipulation of numerals in ways that track underlying arithmetic relations. This mirroring of syntax and semantics is a characteristic of diagrammatic reasoning and explains the cognitive trade-offs that number systems make: For example, the more structure is embedded in a number system, the easier inferences become that exploit this structure — however, this comes at the expense of making the system's representations more constraining.

I explicate two modes of reasoning, representation-internal reasoning and representation-external reasoning, and show that they come with distinct cognitive advantages and costs. The theoretical framework explains why it's easy to use good number notations but hard to discover them. It also explains why there can't be a best notation: Numerical notations work by syntactically embedding target arithmetic structure, but no notation can syntactically embed all, or even most, arithmetical relations. Hence, some trade-offs in representational efficiency are ineliminable. In the second part of this talk, I present a computational model that allows us to study those trade-offs in a quasi-empirical manner. The model compares how a human-like agent learns to perform arithmetic computations in actual and hypothetical number systems. This allows us to investigate the hypothesis that those number systems that we find throughout history strike a balance between learnability and efficiency in computation.

Keywords: numbers, notation, representation, reasoning, cognition, efficiency

Kati Kish Bar-On (Massachusetts Institute of Technology) and Michael Friedman (University of Bonn (Hausdorff School for Mathematics)). ***A Tool or A Collaborator? Rethinking Mathematical Intuition, Authorship, and Practice in the Age of AI.***

Abstract. The increasing integration of AI-based technologies into mathematical practice (such as AlphaGeometry, FunSearch, and Lean) poses a pressing philosophical challenge: how should we reconceive mathematical intuition, authorship, and agency in a landscape increasingly shaped by machine collaboration? This paper addresses that challenge by offering a historically and philosophically grounded analysis of how AI systems are reshaping what it means to "do mathematics." Drawing from four recent case studies (AI-generated solutions in Euclidean geometry, knot theory, combinatorics, and the bunkbed conjecture (Trinh et al. 2024; Miao and Wang 2024; Davies et al., 2021)) we argue that AI programs are not merely tools for computation but co-constitutive agents that transform both the epistemic and social structures of mathematical practice.

At the heart of this transformation is the changing status of the mathematician, a figure traditionally seen as the originator and sole author of mathematical knowledge. As AI-based programs generate solutions that are often beyond human anticipation, mathematicians find themselves collaborating with these technologies. While collaborations between mathematicians and computational tools are not entirely new, the current generation of AI technologies plays a more active role by generating original methods such as proofs, solutions, or examples, and thereby becoming integral to the creative process of producing new mathematical knowledge. Such an integration creates a dissonance with respect to ownership and leadership (as well as a conceptual and definition problem): instead of humans leading the creative process, the AI-based technologies present solutions that mathematicians did not think of and could not have thought of. To what extent, if at all, can these machine-generated proofs be regarded as creative ideas?

Beyond the problem of mathematical creativity, the shift from collaborative, human-centered procedures to more formalized interactions with machines leads to a role reversal in how mathematical intuition is to be understood: from a capacity that may lead mathematicians to new proofs and theorems (Poincaré 1902; Lakatos 1976; Sinclair 2008), it becomes retroactive – something that now needs to be interpreted rather than discovered. This inversion of roles demands a reevaluation of what it means to "do mathematics" and who, or what, is recognized as an active agent or contributor in the discipline.

Most discussions on agency in mathematical practice refer to mathematical activity as human activity, focusing on what people do when they do mathematics (Ferreirós 2016; Hamami 2023), with little substantive exploration of how AI systems that generate original and creative proofs fit into this framework. However, interactions with AI-based programs require mathematicians to develop new skills and adapt to new modes of working. In this new kind of mathematical practice, mathematicians focus on developing and refining AI systems or specialize in interpreting and applying their outputs.

The “working mathematician,” itself another concept in need of revision, is no longer just an individual or a group of mathematicians; instead, mathematical labor is increasingly distributed across human and machine actors, challenging the long-held view that mathematical knowledge is grounded in elementary human activities (Mancosu et al. 2005; De Toffoli and Giardino 2014; Chemla 2015). The paper concludes by proposing a new conceptual framework for understanding mathematics as a hybrid human and machine practice, one that informs how we integrate, interpret, and legitimize AI-generated mathematics across scientific, sociological, and philosophical contexts.

References:

Chemla K. (2015). Proof, generality and the prescription of mathematical action: a nanohistorical approach to communication. Centaurus 57(4):278–300.

Davies A, Veličković P, Buesing L, Blackwell S, Zheng D, Tomašev N, Tanburn R, Battaglia P, Blundell C, Juhász A, Lackenby M, Williamson G, Hassabis D, Kohli P. (2021). Advancing mathematics by guiding human intuition with AI. Nature 600 (7887): 70–74.

De Toffoli, S., & Giardino, V. (2014). An inquiry into the practice of proving in low-dimensional topology. Boston Studies in the History and the Philosophy of Science, 308, 315–336.

Ferreirós J. (2016). Mathematical knowledge and the interplay of practices. Princeton University Press, Princeton.

Hamami, Y. (2023). Agency in Mathematical Practice. In: Sriraman, B. (eds) Handbook of the History and Philosophy of Mathematical Practice. Springer, Cham. https://doi.org/10.1007/978-3-030-19071-2_48-1

Lakatos, Imre (1976) Proofs and refutations: The logic of mathematical discovery. Edited by J. Worrall and E. Zahar. Cambridge, UK: Cambridge University Press.

Mancosu P., Jørgensen K. F., Pedersen S. A. (eds) (2005) Visualization, explanation and reasoning styles in mathematics. Springer, Dordrecht;

Miao, Q., Wang, F. Y. (2024). Artificial Intelligence for Science (AI4S). SpringerBriefs in Service Science. Springer, Cham.

Poincaré, H. (1902). La Science et l’hypothèse. Paris: Flammerion.

Sinclair, N. (2008). Mathematical intuition and the structure of knowledge. Educational Studies in Mathematics, 68(3), 277–293. https://doi.org/10.1007/s10649-008-9111-7

Trinh, Trieu H., Wu, Yuhuai, Le, Quoc V. et al. (2024). “Solving olympiad geometry without human demonstrations”. Nature 625, 476–482. https://doi.org/10.1038/s41586-023-06747-5

Keywords: AI and Mathematical Practice, Mathematical Agency, Mathematical Intuition, Epistemology of Mathematics, Human-Machine Collaboration

Chirine Yasmine Laghjichi (LIPN Université Sorbonne Paris Nord).
The idealized mathematician and the common notion.

Abstract. This talk will aim at comparing two different accounts of the practice of mathematics that refer directly to the notion of the ”common”, or ”community” in the constructivist approach. The first account is the one of the creating subject of L.E.J Brouwer, referring to a sense of mathematical practice that relies on the community of mathematicians. The analysis we are following is the one developed by Kati Kish Bar-on (Kish Bar-On, 2024) taking the opposite view from the usually understood solipsism of Brouwer’s intuitionism relying purely on the individual immediate intuitions. This is compared to the one of G. Kreisel, and his views surrounding his notion of informal rigour (IR), referring directly to the attempts at defining common notions in a scientific setting.

The reason for our focus on these approaches is to get a better grasp at the intertwined notions of ideal in mathematics and the common. To which degree is the former redefined as built upon the later? More generally, if constructivists redefine the mathematical object by stepping back from its conception as an ideal object, how can the community, or the common, share the same subject, or practice? Furthermore, our analysis is strongly linked to the importance their accounts have attributed to redefining the notion of mathematical proof and the developments that led to computer science through its automatization.

We start our reflexion by questioning the role Brouwer attributes to the community and the position it holds in comparison to Hilbert’s program. The strong disagreement between the two scholars is used as a paradigm for understanding the importance Brouwer attributed to ‘social communication processes’ through his contribution in the Significs movement. According to Kati Kish Bar-On, this is a social aspect that goes as deep as in the content.

On the other hand, Kreisel emphasizes in (Kreisel, 1987) the role of the common notion, as an ideal. It is understood through his concept of informal rigor (IR). Church’s Thesis (CT) is taken as a candidate and explored as the object of research having a cultural value. What makes an object of inquiry a candidate for IR isn’t necessarily the possibility of giving a precise definition of the common notion, but the different attempts paving the evolution of the notion.

Lastly, we will show that two candidates for IR, Church’s Thesis as well as Brouwer’s creating subject, that he refers to as the ideal mathematician, are linked through the role of though-processes. According to Kreilse, the creating subject is a justification for a formal system and does not refer to psychological laws but a certain effectiveness. Thus, Brouwer’s creating subject plays an important role in making CT a candidate for Kreisel’s IR.

Keywords: Mathematical practice, Common notion, Creating subject, Informal rigour

Chanwoo Lee (Ajou University). ***Structuralism as an Explanatory Thesis.***

Abstract. Mathematical structuralism is generally taken to be the philosophical position saying that mathematical theories and objects are structural in nature.

Much historical and philosophical scholarship has been devoted to elucidate the concept of 'structural' in mathematical structuralism, but the study on the concept of 'in nature' in mathematical structuralism has been relatively sparse; while, on the one hand, the talk of 'in nature' can be suitably cached out in heavy metaphysical terms (e.g., 'essence'), such a heavy reliance on metaphysics is taken to alienate the practice-oriented aspect of mathematical structuralism, or philosophy of mathematical practice in general.

We argue that such an 'in nature' talk can be more perspicuously understood in terms of explanation, a classic topic in the philosophy of mathematical practice. That is, mathematical structuralism can better be formulated in explanatory terms; mathematical structuralism is a position that takes the structural properties of mathematical concepts and objects to be explanatory. Hence, the rejection of structuralism can be characterized as a rejection of the explanatory power of structural properties in mathematics. Given that structural properties are taken to be universal across the topical boundaries in mathematics, the debate on structuralism dovetails the debate on the ideal of topical purity in mathematics, another major topic in the literature. Hence, such an explanatory formulation of mathematical structuralism elucidates its central idea in a way that connects with the major topics of the philosophy of mathematical practice.

Does this explanatory formulation of mathematical structuralism offer any concrete philosophical benefits? We consider two of such theoretical upshots for mathematical structuralism. First, the explanatory formulation of mathematical structuralism allows us to "offshore" the vexed problem of adjudicating between the eliminative and the non-eliminative version of mathematical structuralism. While this ontological disagreement about the nature of mathematical structure is a substantial problem, the present formulation of mathematical structuralism allows us to view it as a special case of the general debate on the nature of (scientific) explanation. Second, when couched in explanatory terms, mathematical structuralism can be naturally associated with classic cases of mathematical explanation such as those of Galois theory and category theory, which have been analyzed through the structuralist lens in the history of mathematics.

Keywords: Mathematical structuralism, Mathematical explanation, Category theory

Javier Legris (Instituto Interdisciplinario de Economía Política (IIEP), CONICET - UBA). ***Turing on Reasoning in Mathematical Practice: His distinction between intuition and ingenuity.***

Abstract. Alan Turing wrote his PhD thesis during his research stay at Princeton in 1938, with Alonzo Church as his advisor. The dissertation aimed to obtain a complete formalized system for arithmetic in order "to avoid as far as possible the effects of Gödel's theorem," which had proved the incompleteness of arithmetic formalized by finitary means (Turing 1939, p. 99). In his thesis, Turing devised the methodology of "ordinal logics," consisting of constructing successive ordered systems whose completeness can be proved. In this context, Turing introduced the interesting distinction between two procedures (or "faculties") in mathematical reasoning, which he called intuition and ingenuity. Intuition "consists in making spontaneous judgments which are not the result of conscious trains of reasoning" (Turing 1939, p. 214 f.). In contrast, ingenuity consists in representing mathematical knowledge through formulas organized according to logical rules: "The exercise of ingenuity in mathematics consists in aiding the intuition through suitable arrangements of propositions, and perhaps geometrical figures or drawings." (Turing 1939, p. 215)

In this contribution, I will first analyze Turing's characterization of the two aforementioned procedures, intuition and ingenuity, placing them within the framework of his conception of computable methods, which he previously developed through his idea of an ideal machine (nowadays known as "Turing Machine"). Second, I will outline some similarities between Turing's distinction and other perspectives on mathematical proof, particularly the distinction between corollarial reasoning and theorematic reasoning devised by Charles S. Peirce to understand the ways of acquiring genuinely new mathematical knowledge by means of proof (see, inter alia, Peirce NEM IV, p. 49).

This comparison will lead to reflections on computable methods in corollarial reasoning and Peirce's attempts to analyze creative thinking in terms of theorematic reasoning. Finally, I will examine the role of the distinction between intuition and ingenuity in Turing's later characterization of the concept of intelligence, as developed in his seminal paper "Computing Machinery and Intelligence" (Turing 1950). In discussing what he called the "mathematical objection" to his famous test for intelligent behavior, Turing argued that mathematical cognition could reduce to a sequence of "mechanical" procedures arranged in a complex array. In this sense, intuition would reduce to ingenuity. As a conclusion, I will argue that Turing's work in formalized systems and his reflections on mathematical reasoning were guided by a cognitive interest—namely, his idea of mechanical intelligence.

References

Peirce, Charles Sanders. NEM. The New Elements of Mathematics, 4 vols., Carole Eisele (ed.). The Hague: Mouton, 1976. cxxxviii + 2478 pp.

Turing, Alan M. (1939). "Systems of Logic Based on Ordinals". Proceedings of the London Mathematical Society, 2(1), 161–228.

Turing, Alan M. (1950). "Computing Machinery and Intelligence". Mind, LIX (236), 433–460.

Keywords: Mathematical reasoning, Alan Turing, Metamathematics, Mechanical Intelligence, Charles S. Peirce, Mathematical Practice

Jemma Lorenat (Pitzer College). *No marked correlation: an episode in the statistical invasion of the social and medical sciences.*

Abstract. In 1899 the British statistician Alice Lee concluded there was no marked correlation between skull capacity and intelligence in humans. As her calculations were based on a fairly restricted sample, she also called for more data. Over the next several years, Lee and other biometricians working with Karl Pearson at University College London gathered thousands of measurements on school children and college students to compute the relationship between external head measurements and academic performance.

Still there was no marked correlation. Many anatomists and doctors remained vocally skeptical of the statistical conclusions. This talk focuses on Lee's research and its aftermath to elaborate how a variety of actors interpreted the epistemic weight of probabilistic arguments based on the new and laborious technique of correlation. In conversation with the philosophical literature on mathematical hygiene, I will analyze the rhetoric of invasion and disciplinary boundaries used to promote and reject statistics around 1900.

Keywords: history of statistics, interpretations of probability, mathematical hygiene

Danielle Macbeth (Haverford College). *Thinking about Numbers: From Objects to Inquiry.*

Abstract. Beginning with the natural numbers mathematicians over the course of history have come to recognize also, for example, negative and fractional numbers, then irrational numbers, then complex numbers. Philosophers have generally understood this development in terms of the domain extensions required for closure of the inverse operations: the domain of numbers is extended from the natural numbers to include also negative numbers to bring subtraction to closure, extended to fractional numbers to bring division to closure, and to irrational and complex numbers to bring root extraction to closure. But there is also another way to conceptualize what is going on in this corner of mathematical practice. Instead of starting with a two-fold division between the direct arithmetical operations (addition, multiplication, and exponentiation closed under the natural numbers) and their inverses (subtraction, division, and root extraction, which are not so closed), one can start with a threefold division between, first, counting, that is, adding and subtracting conceived as precursors to calculating, then, multiplying and dividing as calculations proper, and finally, exponentiation and root extraction as successor operations to calculating. From this perspective, there are at first no calculations but only practices of counting up and down; at this stage, numbers and operations on them are aspects of one and inseparable.

When we move to calculating with numbers, to multiplying and dividing, we distinguish between numbers and procedures of calculating, procedures that involve counting the numbers themselves. Here we learn to distinguish basic facts about numbers from the derived facts that are the results of calculations. But calculating can take us only so far: there are arithmetical relations among numbers that cannot be revealed in any calculation, most notably, that of the circumference of a circle to its diameter and that of the diagonal of a rectangle to its sides. At the third stage, we turn to reasoning to understand relations among numbers and the numbers they relate. Throughout the model is the practice of traditional Chinese mathematicians. Because there is no symbolization in traditional Chinese mathematical practice, thinking with Chinese mathematicians about numbers not only transforms our thinking about numbers but also sheds new light on the nature and role of symbolizing in mathematical practice.

Keywords: Domain extensions, Traditional Chinese mathematical practice, Nature and function of mathematical symbols

Paolo Mancosu (UC Berkeley). *When formal derivations are not optional: some reflections on derivability claims in first-order vs. second-order theories.*

Abstract. J. Baldwin (2018, 84) pointed out the following:

"Despite its limited expressive power, so far first order logic has been more useful than other logics in formalizing notions of traditional mathematics because the compactness theorem allows stronger applications of the formalization. Thus not only formalization but a particular property of the formalization plays a central role."

The results which Baldwin refers to include compactness and semantical completeness for first-order logic. According to the latter, given a first-order theory formalized in a language L(T), for any sentence A, T derives A iff all models of T are models of A. Notice that proving the right hand side of the equivalence might appeal to "transcendental techniques" (for instance, the models might be models of synthetic geometry and yet the techniques used to establish the truth of a certain sentence in all models are algebraic). In other words, we might be able to know that a derivation of A exists without however having the slightest clue as to how to display one. In this case I will say that the formalized derivation within T is optional (at least with respect to knowing "derivability" in principle). Second-order theories are less commonly employed in contemporary foundations. However, the program of reverse mathematics rests on second-order formalizations of fragments of full second-order arithmetic and recent work in the area of neo-logicism has required the use of second-order theories to capture the Fregean distinction between concepts and objects and to formulate abstraction principles (Hume's principle etc.) mapping concepts into objects. In my presentation I restrict attention to the neologicist arena. As in this context one works with standard semantics, the completeness theorem fails. We might thus be able to prove something about all models of a certain theory T without being able to conclude anything about the (in principle) derivability or refutability of a certain sentence. The interesting situation here is that the "transcendental" (usually set-theoretic) machinery used to establish the semantic claim cannot be easily mimicked to yield derivability results (even in principle) within T and that a careful, and creative, process of formalization is required if one wants to gain a formal derivation within T of a sentence whose truth or falsity in all models is proved by "transcendental" techniques. In this case formal derivations are not optional. In my talk I will contrast the situation in first-order logic (using algebraic and geometrical theories as examples) with that in second-order logic. In the latter case, I will refer to the recent proofs of formal refutability of the conjunction of Hume's principle and the Nuisance principle (Ebels-Duggan 2021), and provide a more detailed discussion of the refutability of abstraction principles satisfying the "part-whole" constraint (Mancosu & Siskind, 2019).

Bibliography

Badwin, J., 2018, Model Theory and the Philosophy of Mathematical Practice. Formalization without Foundationalism, Cambridge University Press.

Mancosu, P. and Siskind, B., 2019, "Neologicist Foundations: Inconsistent abstraction principles and part-whole", in Mras, G. et al. (eds.), Philosophy of Logic and Mathematics: Proceedings of the 41st International Wittgenstein Symposium, De Gruyter, 215–248.

Ebels-Duggan, S., 2021, “Deductive Cardinality Results and Nuisance-Like Principles”, The Review of Symbolic Logic, 14, 592-623.

Keywords: formalized theories, completeness theorem, transcendental methods, neologicism

Oliver Marshall (UNAM (Visitor)). *The Logical and Philosophical Origins of λ-Calculi.*

Abstract. Today, we are used to thinking of λ-calculi as tools for theorizing in philosophy, computer science and linguistics, and as objects of mathematical study in their own right. However, they originate from Alonzo Church’s early (1932-35) attempt to provide a foundation for mathematics in the style of Hilbert. While work in this tradition by Hilbert, Bernays, Gödel and Herbrand has been covered thoroughly, Church’s contributions, as well as critiques by Kleene and Rosser, have not received enough attention. In this talk, these contributions take center stage.

Studying Church is especially illuminating because he discusses the philosophical outlook that he weaves into his presentation of formal logic and formalized mathematics. This is most interesting in the present case because the logic in which Church embeds λ is neither classical nor intuitionistic. Church’s foundational proposal is also of note because functions are taken to be the basic notion rather than classes, sets, or natural numbers. Another point of interest is that although, as I will explain, the project was dogged by the paradoxes of Russell, Richard, and Curry, as well as by Gödel’s theorems, Church’s final proposal (1935) seems to be consistent due to its unusual treatment of implication. This idea, which bears a certain similarity to one from Martin-Löff type theory, as well as to free logic, was subsequently taken up by Myhill (1975) but has never been spelled out in full detail.

Church’s work and reflection on the problems that arose from it led quickly to such major developments as the isolation of the pure untyped λ-calculi, Kleene’s discovery of recursive function theory, and Church’s proof that the Entscheidungsproblem is unsolvable.

I will discuss some of this history, which is a case study of the feedback that occurs between mathematics, philosophy and logic, when grand philosophical projects not only inspire the discovery of theorems that refute them, but also inspire the development of ideas with unexpected applications. Then I will highlight some issues that are (or should be) the subject of interdisciplinary work among contemporary mathematicians, logicians, computer scientists and philosophers.

Keywords: λ-calculi, formalism, paradox

Koji Mineshima (Keio University) and Ryota Akiyoshi (The University of Electro-Communications). *Concessive Connectives and the Structure of Informal Proofs.*

Abstract. While much attention has been paid to the structure and automation of formal mathematical proofs, the linguistic and pragmatic features of informal proofs and their relation to formal proofs remain relatively underexplored. One notable phenomenon in informal mathematical proofs is the frequent use of concessive connectives such as "but". This talk investigates the roles these expressions play in informal proofs, drawing on insights from natural language semantics and pragmatics, the philosophy of mathematics literature on informal proofs, and the proof libraries of interactive theorem provers.

The role of concessives such as "but" and "although" in natural language has been the subject of extensive analysis by both philosophers and linguists. A widely shared view holds that the use of "but" involves signaling contrast or rhetorical expectation violation. Frege (1879), for instance, famously noted that his Begriffsschrift has no expression corresponding to the difference between "and" and "but", and that "but" serves "to hint that what follows is different from what might at first be supposed." Similarly, Quine (1941/1980) claimed that "although" is used "only for rhetorical purposes." In the linguistics literature on discourse pragmatics, "but" is often analyzed as triggering a "denial of expectation" inference (Kehler 2002).

For example, the sentence "John is poor, but he is happy" presupposes the generalization "If John is poor, then normally he is not happy." Schematically, the structure "P but Q" presupposes the inference "If P, then normally not Q." Such analyses involving denial of expectation are also widely adopted in corpus-based classifications of discourse relations (Zufferey and Degand 2024).

In this talk, we first argue that this standard analysis does not straightforwardly apply to informal mathematical proofs. In typical cases, the role of "but" is not to signal contrast or rhetorical expectation violation, but rather to mark a crucial inferential step in the derivation of the conclusion. In our view, concessive expressions such as "but" serve as important cues for expressing and reconstructing proof structures.

To examine the specific roles that concessive connectives play within the structure of informal proofs, we focus on a representative set of informal proofs from elementary logic and set theory textbooks, alongside their corresponding formal proofs written in the Lean proof assistant. We classify the pragmatic roles of each occurrence and analyze how they align with specific steps in the formal proofs. By comparing informal and formal proofs, we investigate how concessive connectives contribute to producing proofs with an internal structure that is more accessible to human understanding. We also examine and compare several approaches to specifying the structure of informal proofs, as discussed in the philosophy of mathematical practice literature (Tanswell 2024). This study is part of a broader interdisciplinary project that seeks to clarify what counts as natural and explainable proof in an era where the prospect of widespread use of computer-assisted mathematical proofs is beginning to emerge.

Keywords: Informal Proofs, Concessive Connectives, Proof Assistants, Discourse Pragmatics

Alberto Naibo (Université Paris 1 Panthéon-Sorbonne, Institut d'Histoire et de Philosophie des Sciences et des Techniques (UMR 8590)) and Thomas Seiller (CNRS). ***Mathematical proofs as algorithms, formalized proofs as programs.***

Abstract. We aim to compare the proofs typically found in mathematical textbooks and articles (referred as "prose proofs" by Heather Macbeth, and which we simply call "mathematical proofs" here) with those written in regimented formal languages, such as those used in proof assistants (which we refer to as "formalized proofs"). We argue that the relationship between these two types of proofs can be understood by analogy with the relationship between algorithms and programs. Mathematical proofs, like algorithms, are expressed in a relatively compact form, as their primary aim is to be communicated to other mathematicians. Just as an algorithm consists of a structured set of instructions to be understood and followed by an agent to accomplish a task, a mathematical proof provides a general structure to be understood and followed to reach a conclusion. This structure is presented at a high level of abstraction, pointing to the lemmas that have to be used and indicating the proof strategies to be applied (e.g., "by induction", "by contraposition", etc.). Both the algorithm instructions and the proof strategies can be analyzed into several more basic operations: this analysis depends on (the background knowledge of) the agent to which the algorithm or the proof is addressed, meaning they are not fixed once and for all (e.g., the level of detail in a proof may vary between a textbook and a research article). On the contrary, formalized proofs, like programs, are bound to a specific language and a fixed set of rules and operations that implement the general strategies found in mathematical proofs. In the case of programs, fixing a set of primitive operations and particular representations of the data allows one to effectively execute them (in particular, computational costs can be evaluated at the level of the program, not at the level of the algorithm). Similarly, formalized proofs can be computationally verified (i.e., their correctness can be mechanically checked) and they allow for a precise evaluation of the axioms and rules (i.e., the theory) needed to prove a certain theorem. According to our view, mathematical and formalized proofs are not in opposition but they are interdependent and complement each other.

They meet different epistemic goals: formalized proofs correspond to concrete implementations of mathematical proofs, allowing for verifiability, while mathematical proofs provide the specification (the blueprint) of formalized proofs, allowing for communicability and understandability. This explains why, contrary to a certain strand of contemporary debate, we avoid speaking of "informal" versus "formal" proofs. We want to avoid the idea that one kind of proofs has primacy over the other, so that the latter would simply be lacking some properties of the former.

Keywords: Mathematical proofs, Formalized proofs, Algorithms and programs

Jacob Parish (UC Berkeley). ***Non-standard analysis, divergent series and an application to Grosseteste's infinite summations in De Luce.***

Abstract. In classical analysis, divergent series such as 1 + 2 + 3 + … lack defined values, and any attempt to perform arithmetic operations with them is deemed invalid. Nevertheless, this has not prevented mathematicians and philosophers from expressing intuitions about the relative sizes and arithmetic relationships between such infinite sums. One of the earliest examples appears in Robert Grosseteste's 13th century treatise De Luce, written several centuries before the formal development of the notion of a series. Grosseteste articulated several claims about infinite aggregatios of numbers, asserting, for example, that the sum of all natural numbers exceeds the sum of all even numbers by the sum of the odd numbers, and that the sum of all numbers obtained by successive doublings is twice the sum of their corresponding halves. From the perspective of classical analysis, these statements are meaningless, as they involve operations on divergent series. Nevertheless, Grosseteste's reasoning reflects an intuition about infinite sums and the relations they ought to satisfy, which merits further investigation.

I will explore several modern approaches to divergent series which use non-standard analysis, and the extent to which they can be used to formalize such intuitions about infinite sums.

The first two approaches examined are due to Bartlett, Gaastra, and Nemati (2020), and to Bellomo and Massas (2023), and each provides a method of assigning infinite hyperreal numbers to classically divergent series. A third approach, due to Kanovei and Reeken (1995), while less directly applicable to Grosseteste's specific claims, provides a way to understand other mathematical techniques for evaluating divergent series, particularly those of Euler. The first two approaches will allow us to interpret Grosseteste's examples in a mathematically consistent way, although some reinterpretation of the series is required. For instance, while it is not true in these frameworks that (1 + 2 + 3 + …) - (2 + 4 + 8 + …) = 1 + 3 + 5 + …, it will be true that (1 + 2 + 3 + …) - (0 + 2 + 0 + 4 + …) = 1 + 0 + 3 + 0 + …

Keywords: nonstandard analysis, divergent series, robert grosseteste

Mangesh Patwardhan (National Insurance Academy, Pune, India). ***Sets, Models and Set-theoretic Practice.***

Abstract. What are the fundamental objects of study in set-theoretic practice? Well, sets themselves! At least, that has arguably been the default position. The pervasive independence phenomenon in set theory has put pressure on this. In particular, with Paul Cohen's introduction of the technique of forcing, there has been a proliferation of models of ZFC and its extensions. In this context, Joel Hamkins forcefully argues that now the fundamental objects of set-theoretic research are models of set theory, rather than sets in the standard (class) model – the iterative hierarchy V. He compares the current situation in set theory with the emergence of several alternate geometries in the nineteenth century. Carolin Antos draws on Penelope Maddy's naturalistic and practice-based approach to analyze this Hamkinsian claim. Based on the contemporary practice of forcing, she claims that models have (also) become the fundamental entities in set-theoretic research. The model-based approaches to forcing (countable transitive models of a finite fragment of ZFC or its variants (ctm) and the Boolean-valued models (bvm)) are now common and seen to be mathematically fruitful.

In contrast, Cohen's syntactic approach never had the same importance for practice, nor has proved to be fruitful. She notes that parts of set theory may still take place in the standard model and so both sets and models may be treated as fundamental entities, albeit in different contexts of set-theoretic practice.

I argue that contra Antos, ctm and bvm approaches do not provide support for models-as-fundamental entities (mafe) argument and actually undercut it. These approaches arguably treat these models in a purely instrumentalist way, rather than as adequate and legitimate models of set theory. I offer an alternative justification for mafe. I agree with Hamkins that the undecidability of certain celebrated set-theoretic hypotheses such as CH from ZFC and its extensions; and the metamathematical issues about forcing provide a robust justification for adopting this approach. However, I present my argument in the spirit of Timothy Williamson's epistemicist position. This position treats vagueness as the limits on our capacity to acquire exact knowledge. As Williamson puts it, the inexactness here is in the knowledge, not in the object about which it was acquired. He employs the principle of "margin for error" in the context of the limitations on our cognitive capacities. I draw on this principle in the set-theoretic context in the backdrop of the limitations on our proof-theoretic capacities. I argue that this formulation also avoids an appeal to the (rather controversial) Gödelian mathematical platonism implicit in Hamkins' defence of his position. My formulation has affinities with Sharon Berry's "pluralism by levelling up" argument in a related context.

Finally, I consider recent positions by Penelope Maddy and Neil Barton wherein they attempt to motivate and defend (in effect) the sets-as-fundamental entities position. I argue that their arguments effectively amount to and strengthen the case for mafe. In the end, I claim that in contemporary set-theoretic practice, it is models all the way!

Keywords: Fundamental entities in set-theoretic practice, Independence phenomenon in set theory, Epistemicism and set theory

Jean-Charles Pelland (University of Bergen). *Compositionality beyond bases.*

Abstract. Compositionality beyond bases

While mathematicians specialize in a wide variety of sub-disciplines, each one of them started their mathematical career the same way: learning how to use numerals. For children learning to manipulate numbers in developed countries, this means mastering a numeration system equipped with a base. The concept of the base of a numeration system has been productively used to analyse and categorize both numerical notations (Chrisomalis 2010) and lexical numerals (Greenberg 1978; Comrie 2005). However, the way many linguists have framed the concept of a base deviates significantly from how bases are defined in mathematical notations. While it has not necessarily hindered intra-disciplinary progress, this conceptual clash slows inter-disciplinary research on important questions about our numerical abilities, including investigation into their potential origins in bodybased numerals, and the shared cognitive resources recruited in numerical cognition across representational formats.

This talk has two main objectives. First, to show that the conflicting way the concept of base has been used in the study of notations and of lexical numerals is not well suited for a general investigation into the compositionality of representations of numbers across representational formats. Second, to lay down the roots of a conceptual framework capable of fulfilling this role. To achieve these goals, the talk unfolds as follows. I start by surveying how bases of mathematical notations are usually defined (Widom & Schlimm 2012; Chrisomalis 2010, 2004), in order to show how this notational approach lumps many different systems, both linguistic and bodybased, under the unhelpful 'not-a-base' label. I then survey how some linguists have defined bases (Comrie 2005; Barlow 2025; Conant 1896) in order to show how this approach also has a lumping problem, but under the 'base' label.

Having shown the limitations of these approaches, I analyze examples of tallies so as to pump intuitions about what sort of features may be important to capture in a format-independent conception of bases, focusing especially on shifts in counting units, and the importance of chunking mechanisms (Gobet et al. 2001; Cowan 2005; Dotan & Brutman 2022) to form counting units out of pluralities. I then illustrate the value of using counting units as a core concept with which to analyse numeration systems by comparing how Indo-Arabic and Roman numerals individuate labels for their bases, showing how only the former does so via a recursive function. This leads to a cognitive definition of base in terms of powers of a number as counting units, and to the introduction of a broader compositional tool that I call anchors. I end the talk by introducing the basics of anchors and illustrating their potential as analytical tools by applying them to concrete examples.

Keywords: Numerals, Base, Chunking, Numerical Cognition

Christopher Pincock (Ohio State). *Mathematics and scientific change in the eighteenth century.*

Abstract. The most common account of how new mathematical knowledge leads to new scientific knowledge is based on the assumption that mathematics enables scientists to recognize more clearly the implications of their non-mathematical assumptions. In this paper I challenge this inferentialist account and develop an alternative that focuses on the changing character of scientific problems. In some cases, I argue, new mathematical knowledge encourages a substantial reconceptualization of a scientific problem. One way for this to occur is that new mathematical knowledge indicates new and fruitful ways of relating mathematics to the natural world. When this occurs, it becomes clear how innovations in mathematics can have profound effects on scientific knowledge.

To illustrate my problem-based proposal, I consider some of the changes in the character of two families of problems in the work of Newton and in the post-Newtonian period leading up to Fourier's Analytical Theory of Heat (1822). Cartesians, Newton and post-Newtonians such as Euler and Clairaut took both the orbit of the moon and the tides as significant scientific problems. While Newton made some progress addressing both families of problems, his accounts were widely seen to be inadequate and potential sources of alternatives to the law of universal gravitation. I discuss how post-Newtonians not only used new areas of mathematics to address these problems, but how these uses involved a different understanding of the relationship between mathematics and the natural world. One symptom of these changes is that while Newton restricted "rational mechanics" to "the science … of the motions that result from any forces whatever and of the forces that are required for any motions whatever" (1686 Preface to Principia), Fourier boldly declared that "mathematical analysis is as extensive as nature itself" (1822 Preliminary Discourse to Analytical Theory). This vast increase in the scope of applied mathematics went along with a new understanding of how mathematics should be related to nature.

One implication of the problem-based approach to mathematization is that philosophers should recognize the more historicist and contextual aspects of the scientific knowledge that results. I conclude by considering whether or not these aspects mandate a revised conception of knowledge, as in the work of Kuhn, Massimi or Chang.

Keywords: applied mathematics, rational mechanics, analytical mechanics

Erich Reck (University of California at Riverside). *Completeness as a Value in Mathematics.*

Abstract. The role of mathematical values, goals, or ideals has started to draw considerable attention in the philosophy of mathematical practice, as illustrated by purity, depth, and being explanatory. In this talk I will consider another case, namely the completeness of mathematical theories seen as a similar value. Building on previous work of mine, I will start with a brief summary of the early history of the notion of completeness in modern mathematics, from Dedekind, Hilbert, and Veblen through Fraenkel's and Carnap's related works on to Gödel's incompleteness theorems from the early 1930s. In doing so,

I will highlight several aspects: (i) three related ways in which the completeness of mathematical theories can be understood, as became clear along the way, namely as syntactic completeness, semantic completeness, and categoricity; (ii) the manner in which the completeness of theories, in any of these senses, relates to the completeness of logical deduction systems in the background; and (iii) how Gödel's results affects both sides. It is natural to think, and some philosophers have argued explicitly, that with these results completeness has to be given up as a mathematical goal. Edmund Husserl provides an early illustration of this response, as shown in recent work by Mirja Hartimo.

But as I will argue in response, the situation is richer and more complicated. This can be illustrated with five different kinds of responses to Gödel's results: (a) in model theory, these results led to the focus on first-order logic and the replacement of completeness by weaker variants, such as categoricity in some cardinality. (b) In work on the foundations of mathematics, an important question became whether or not one can replace the original, meta-logically inspired Gödel sentence by more "natural" sentences that are also independent, with the Paris-Harington theorem as illustration. (c) In set theory, Gödelian incompleteness was complemented by other independence results, leading to the search for new axioms. (d) In much of mathematical practice, in contrast, categoricity in something close to its original form, and with some form of higher-order logic used as the informal framework, has continued to play a role. And (e) the latter includes category theory, where related results involving homomorphisms and isomorphism are central. I will conclude the talk by exploring whether this difference in responses, in particular the partial retainment of categoricity as a goal in mathematical practice, can be explained in terms of some notion or notions of "internal categoricity/completeness" that are at play.

Keywords: completeness, categoricity, mathematical theories, values in mathematics

Patrick Ryan (New York University). ***A Renaissance of Empiricism: The Case of Mathematical Infinity.***

Abstract. In this talk, I will investigate the relationship between two central topics in the philosophy of mathematics:(i) our intuitive criteria (and attendant formalizations) for "measuring" infinite sets and (ii) the nature of justification in mathematical practice.

We have at least three intuitive criteria on offer. Let us call the first criterion Cantor's Principle (CP):two infinite sets A and B have the same "size" if and only if their elements can be put into 1-1 correspondence. This criterion is formalized by Cantorian cardinalities and has been examined in great detail. An alternative criterion, Part-Whole (PW), says: if A is a proper subset of B, then the size of A should be strictly less than the size of B. (PW) has been adequately formalized only recently by the theory of numerosities. The third and final criterion for measuring infinite sets involves a Frequency (FR) intuition: if infinite sets A and B occur "equally often" in an ambient set C, then A and B have the same size. (FR) finds expression in the number-theoretic notion of density.

Recently, there has been an explosion of philosophical work surrounding the theory of numerosities. However, a limitation of studying numerosities is that the theory is still in its infancy, and thus its implications for mathematics as a whole are unclear. On the other hand, densities have been a central tool in mathematics (especially number theory) for over a century. Indeed, there is a rich

collection of results that inextricably involve densities, and these results offer the philosopher of mathematics much data for reflection. In particular, an examination of the uses of densities in mathematical practice can shed light on the nature of justification in mathematics.

There is a long and venerable tradition in the philosophy of mathematics that views mathematics, especially in its canons of justification, as continuous with the natural sciences. Following Lakatos, let us call this conception of mathematics "quasi-empiricism." The central tenet of quasi-empiricism is that the acceptance

of axioms for mathematics follows not from criteria of certainty but rather from the success of the axioms in deriving and systematizing basic mathematical truths. Thus, axioms in mathematics discharge a similar epistemic role as physical laws that systematize empirical phenomena. However, I wish to emphasize that it is not only in foundational work that we witness the quasi-empirical character of mathematics. It is, in fact,even more pervasive in the selection and justification of definitions and concepts in first-order mathematical practice. I will defend this thesis as applied to densities in number theory. In particular, by examining results of probabilistic number theory, beginning with the path-breaking work of Hardy and Ramanujan, I will show that density is the only way to validate

deeply-seated intuitions or "basic statements" about the structure of the natural numbers. It is by virtue of this fact that (FR) and densities have had wide currency as central concepts in number theory, and thus demonstrates the quasi-empirical character of mathematics in an interesting case.

Keywords: Mathematical Infinity, Justification in Mathematics, Number Theory

Andrea Sereni (IUSS Pavia). *The Ideal Route to Arithmetic.*

Abstract. An essential aim of the Neo-Fregean program is to show, based on Frege's Theorem, that the basic laws of arithmetic are knowable a priori (and analytic in Frege's sense). If HP is knowable a priori and the program succeeds, it gives us an ideal, a priori route to arithmetical knowledge – or, at least, to knowledge of Frege Arithmetic (FA). A question can be raised, and has been raised Cf. Heck [2000], Wright [2016]), of how the ideal route to knowledge of FA can inform, or even match, our ordinary or pre-theoretical arithmetical knowledge. Recently, this has been referred to as The Neglected Question. The question bifurcates:

1. The Matching Problem: is (knowledge of) Frege-Arithmetic just (knowledge of) arithmetic?

2. The Epistemic Improvement Problem: assuming a positive answer to (1), does the ideal route to Frege-Arithmetic provide epistemic support/improvement to the epistemic status of pre-theoretical arithmetical beliefs?

Here we will focus on (1), assuming that:

• it is a precondition to answering (2), and hence underlies any attempts to tackled the NQ

• it has indeed been neglected.

There a number of preliminary points for concern:

• it is not clear what "actual", "ordinary" or "pre-theoretic" arithmetic comes to, nor what would be the target of the sought-for explanation:

– psychological or physiological process of belief-acquisition?

– meaning attributed by naive speakers to arithmetical propositions?

– elementary patterns of use of arithmetical propositions, including basic proof procedures?

– pre-theoretical (doxastic) justificatory patterns for arithmetical propositions?

• the conflation of "actual" or "pre-theoretic" with "ordinary" or "elementary" is dubious: foundational projects may affect, or ask for revisions or qualification,

of much more sophisticated conceptions than those of naive speakers (to wit: the entire pars denstruens of Frege [1884]).

• whatever the answer, it's not clear what the purposes of a foundational project should be with respect to pre-theoretical knowledge.

– to reveal, hermeneutically, what we meant all along with arithmetical propositions (even if unaware)

– to replace our pre-theoretical beliefs with different, homophonic, beliefs

– to fix our justificatory (doxastic) procedures with safer (propositional) justification?the ideal route to arithmetic

– to certify that our pre-theoretical arithmetical beliefs were epistemically safe enough to begin with?

Our aims in this talk will be:

1. To discuss ways in which the NQ has not entirely been neglected by neo-Fregeans, by suggesting they are not effective enough nonetheless:

• Frege's Constraint and Gentzen-style interpretation of HP seem to go in the direction of contributing an answer to the NQ

2. To discuss limitations of the epistemic setting of the Neologicist Program which seem to prevent answering the Matching Problem

• Hale and Wright's conception of Reconstructive Epistemology and a tension between analysis and rational reconstruction

3. to suggest ways in which the Neo-Fregean ideal route could be understood to address the Matching Problem and NQ

• we'll gesture at a Carnapian understanding of the ideal route in terms of explication.

The overarching aim is to throw light on how the epistemology of foundational programs like Neologicism relate to the justification of aritthmetial propositions as employed in actual mathematical practice.

References

Gottlob Frege. Die Grundlagen der Arithmetik: eine logische mathematische Unter- suchung über den Begriff der Zahl. Koebner. English translation by J. Austin, The

Foundations of Arithmetic. A Logico-Mathematical Enquiry into the Concept of Number, Blackwell, Oxford, 1950/1953, Breslau, 1884.

Richard Kimberly Heck. Cardinality, Counting, and Equinumerosity. Notre Dame Journal of Formal Logic, 41(3):187–209, 2000.

Crispin Wright. Abstraction and Epistemic Entitlement: On the Epistemological Status of Hume's Principle. In P. Ebert and M. Rossberg, editors, Abstractionism, pages 161–85. Oxford University Press, 2016.

Keywords: Neologicism, Foundations of arithmetic, Explication

Samuel Stevens (University of California, Berkeley). ***Plato and the Practice of Mathematics.***

Abstract. A central puzzle in the study of Plato's philosophy of mathematics is to explain what Plato did not say. Aristotle tells us that Plato held mathematical objects to be ontologically intermediate between sensible and Forms because mathematical objects, like forms, are eternal and unmoving but also, like sensibles, multiple (Metaph. 987b14–18, 1028b19–21). But Plato never gives this argument. Indeed, he tells us nothing direct about either the ontology or logical structure of mathematics. Into this silence, nearly every logically possible interpretation has been suggested, yet none are able to ground an interpretation firmly in the text. Ultimately, scholars either have to convict Plato of incoherence, argue that he withheld his true doctrines, or suggest that he intentionally underdetermined a central part of his argument to spark us to inquire for ourselves.

I will suggest that this difficulty arises because we are asking the wrong question. If we do not assume that Plato was concerned with mathematical ontology but instead take him to be doing something similar to what has recently come under the name "the philosophy of mathematical practice," then we will see his explicit arguments are coherent and complete. I will develop this through a reading of the Divided Line in Plato's Republic. His point there is primarily epistemological. He is interested in the way that we can come to know, the process of being led to truth. Geometry enters as an example to show how this process can be successful. Key here, I argue, are a series of characteristics of ancient geometry that have been lost in modern mathematical practice. Ancient geometry proved propositions by constructing a diagram. As others have shown, these diagrams were often non-metrical. They were drawn to suggest the relationships between lines rather than accurately represent them.

This was sometimes a technological necessity—the techniques of perspective were still rudimentary, constructing complex figures with only a straightedge and compass can be overwhelmingly time-consuming, and much ancient mathematical work was done with chalk or in sand—but this disregard for exact construction seems to have been a broader feature of diagrams that could have been constructed exactly. Further, the diagram was accompanied by the text of a proof, but neither diagram nor text alone contain all of the logical information required for the proof. It is only the conjunction of the two that allows the student to understand the proof. The consequence of this, and what geometry such a useful example for Plato, is that it is impossible for the student to reduce the proof to the diagram. Working through the proof forces one to recognize that the true object of the proof is something incorporeal, abstract, and only partially represented in the image. This is, as is widely recognized, precisely the dynamic that Plato suggests holds more generally between Forms and the images of Forms. Geometry is a natural way for Plato to display this dynamic.

Keywords: Plato, Ancient Geometry, Epistemology

Jonathan Tanaka (University of California, Berkeley). ***Aristotelian Geometrical Monism.***

Abstract. That astronomical or kinematic facts could decide the question of whether there is a unique, maximal, true geometrical theory sounds, prima facie, absurd. From the standpoint of contemporary mathematical practice, geometries are pure, axiomatically individuated mathematical theories with truth conditions independent of the de facto curvature structure of the world, let alone its absolute metrical properties. Even on an Aristotelian conception – an in rebus realist ontology – geometry is the science of perceptible objects inquantum (ᾗ) they are continuous and quantitative (e.g., Metaphysics Κ III, IV); thus, the domain over which geometry theoretically legislates is in one sense just the range of possible physical objects themselves, but abstracted from their metrical properties, including the metrical properties of the cosmos in which they are spatially embedded.

Despite this, I infer and evaluate what I argue is exactly this rather surprising corollary of Aristotle's finitism concerning astronomy and locomotive κίνησις (change), given his conception of geometrical practice – that geometrical monism is true. In particular, the uniquely true geometry on Aristotle's astronomical assumptions is in fact a kind of Euclidean geometry. The discussion illuminates an underappreciated way in which, conditioning on a rather conservative mathematical ontology, it is precisely an optimism about mathematical practice that can logically constrain the space of genuine mathematical theories. Thus, it highlights underappreciated ways in which conservative positions in debates over mathematical pluralism may be motivated.

Hence, in this paper I clarify Aristotle's geometrical ontology and his doctrine of ἀφαίρεσις (abstraction), yielding a new exposition of his conception of geometrical practice. This in turn suggests a novel interpretation of his 'solution' in Physics Γ to the problem of reconciling face-value geometrical practice with his mathematical finitism. Using Aristotle's subtle understanding of geometrical practice, I argue that a central modification of a traditional interpretation of this passage is correct, and that certain novel interpretive options, such as the recent "converse increasing" approach of Ugaglia (2016), are ultimately untenable for the Aristotelian. The surprising and unrecognized corollary to this newly-interpreted solution – that if the universe is necessarily finite, non-Euclidean geometries must necessarily be devoid of the content necessary for science par excellence – subsequently follows without ingenious argument. In overviewing Aristotelian astronomy and how, modulo some analytic reasoning, it entails finitism about perceptible magnitudes, the corollary's antecedent is thus discharged, yielding geometrical monism. However, none of this is to anachronistically claim that Aristotle would claim the parallel postulate as a geometrical axiom. Rather, the discussion reveals an interesting distinction between two kinds of ways that geometrical possibility can be limited – (a) the positing of further axioms or common notions, and (b) the metatheoretical commitment that certain interpretations of geometrical practice are uniformly possible. It is by geometrical limitation in virtue of metatheoretical commitment to an interpretation of mathematical practice, rather than axiomatic commitment, that Aristotelian finitism is rendered consistent with the face-value geometrical practice of his day.

Keywords: axiomatic geometry, Aristotelian geometry, mathematical pluralism, mathematical ontology, finitism

Jamie Tappenden (Philosophy, University of Michigan). ***Definition, vagueness and conceptual development in mathematical practice.***

Abstract. Tyler Burge, taking a cue from Fregean remarks on ``partial grasp" of concepts, has perceptively explored concepts whose original definitions are in some way or other incomplete, requiring additional development before they are adequately understood. Burge also applies the framework to mathematical concepts, in particular the relationship between Newton's and Weierstrass' definition of the derivative. This example has been widely discussed, both critically and supportively, notably by Christopher Peacocke, Georges Rey, Sheldon Smith and Sebastien Gandon.

In this paper I'll further explore Burge's insights in connection with a different example of the rigorisation of an initially partially grasped concept – the genus of a curve / surface. The case begins with what we can now recognise as an early, obscure appearance of the genus, in what is sometimes called Abel's Addition Theorem for Abelian functions. The concept was then clarified and sharpened in a variety of different but equivalent ways in subsequent research. The paper will principally discuss Bernhard Riemann's thesis and subsequent work, where the genus appears as what we would now call a topological fact about the connectivity of surfaces, and its appearance as what we would now call a concept of algebraic geometry in the work of Alfred Clebsch and his students. The respects in which the later, more precise versions of the genus capture a concept that is in some sense present in the earlier work will turn out to be quite involved, turning on both the underlying mathematical facts and the methodological commitments and intentions of the mathematicians and schools involved.

Keywords: Concept, Definition, Partial Grasp, Vagueness, Riemann, Abel, Clebsch, Burge

Karolina Tytko (Independend researcher), Jessica Carter (Centre for Science Studies & Department of Mathematics, Aarhus University, Aarhus, Denmark), and Karol Wapniarski (Faculty of Psychology and Cognitive Science, Adam Mickiewicz University, Poznań), ***Experiments in Logic and the Role of Diagrammatic Representations: A Case Study of Aristotelian Syllogistic Proof Charts.***

Abstract. In the light of the recent turn in the philosophy of science towards philosophical analysis of scientific practices – a trend including also philosophy of mathematics and logic (Dutilh Novaes 2012; Giardino 2017; Carter 2019; Tytko et al. 2023) – we present a particular case study on charts used in connection with syllogistic proofs, illustrating the experimental use of such charts in the process of discovering logical results.

The related case concerns Aristotelian syllogistic proofs. Particularly, we describe how the collection of all computed alternative direct and indirect syllogistic proofs on specific charts have allowed to view the system of syllogistic logic from a different angle and hence to discover properties of it which were earlier overlooked (Wapniarski and Urbański 2025).

Drawing from a characterization of mathematical experiments (Schlimm and Fernández González 2020), we propose to extend this account to logic and to view the described case as a kind of experiment in logic (or logical experiment). In particular, we treat the charts in question and the properties of syllogisms read from them as components of the experimental process with an outcome and interpretation (conclusion). As tools in this experiment, we consider the charts and diagrams, as well as the proof assistant (Isabelle/HOL) used for a formal verification of the results pertaining to direct proofs (Koutsoukou-Argyraki and Wapniarski 2025).

In contrast with the experimental setup described in (Schlimm and Fernández González 2020), the setup in this case includes presenting the results on charts – arranged in this form, the results revealed certain patterns, and thus the form of representation played a crucial role for the discovery of new properties. In this context, we argue that the case-study exemplifies the phenomenon of a "free ride" as characterized in (Shimojima 2015) and (Carter 2021), drawing attention to the importance of visual thinking in logic.

Carter, J. 2019. „Philosophy of Mathematical Practice – Motivations, Themes and Prospects". Philosophia Mathematica 27(1):1–32. https://doi.org/10.1093/philmat/nkz002.

Carter, J. 2021. „"Free rides" in Mathematics". Synthese 199, 10475–10498. https://doi.org/10.1007/s11229-021-03255-9.

Dutilh Novaes, C. (2012). Towards a Practice-based Philosophy of Logic: Formal Languages as a Case Study: From Practice to Results in Logic and Mathematics. Philosophia Scientiae, 16(1),71–102.

Giardino, V. 2017. „The Practical Turn in Philosophy of Mathematics: A Portrait of a Young Discipline". Phenomenology and Mind, 12,18–28.

Giardino, V. 2018. „Manipulative imagination: How to move things around in mathematics". THEORIA. An International Journal for Theory, History and Foundations of Science 33(2),345–360.

Koutsoukou-Argyraki, A. and Wapniarski, K. (2025). A Formalisation of Aristotle's Assertoric Syllogistic in Isabelle/HOL. Topoi – An International Review of Philosophy [paper accepted for publication, preprint available at: https://www.researchgate.net/publication/386107729_A_Formalisation_of_Aristotle's_Assertoric_Syllogistic_in_IsabelleHOL]

Schlimm, D. and Fernández González, J. 2020. „Mathematical Experiments on Paper and Computer". In: Handbook of the History and Philosophy of Mathematical Practice, Bharath Sriraman (ed.), 1–20. Cham: Springer International Publishing.

Shimojima, A. 2015. Semantic Properties of Diagrams and Their Cognitive Potentials. Stanford, CA: CSLI Publications.

Tytko, K., Mangraviti, S., Rylko, N. and Karlygash, N. B. 2023. „New epistemology of mathematics and formal sciences in the age of AI. Critical concept kinds and diversity of mental representations". Edukacja Biologiczna i Środowiskowa 2(80), DOI: 10.24131/3247.230207 .

Wapniarski, K. and Urbański, M. 2025. „Aristotelian Syllogistic more apagogico demonstrata: identifying minimal sets of inference rules for indirectly proving all the syllogistic moods" [under review in Logica Universalis].

Keywords: Free rides, visual representations, Aristotelian syllogistic, experiments

Susan Vineberg (Wayne State University). *Depth and Understanding in the Development of Mathematics.*

Abstract. There has been considerable philosophical discussion in recent years about mathematical explanation, accompanied by the ongoing development of various accounts. At the same time, following Mancosu, there has been a call to produce case studies that can be used as evidence both for testing a proposed theory of explanation and as background data against which new accounts might be developed and older ones refined. This paper examines a recent case study and then uses it to observe some key differences between various treatments of mathematical explanation.

The first part of the paper considers Baldwin's insightful study of proofs of completeness, in which he argues that Steiner's account of mathematical explanation can be modified by expanding the idea of a characterizing property to count Henkin's proof of the completeness theorem as explanatory. I endorse both Baldwin's assessment of the significance of Henkin's proof in the development of model theory and the idea that the proof may be said to be explanatory.

However, it is not so clear that the reasons that it is explanatory are captured by extending Steiner's notion of a characterizing property of an object to the characterizing property of a proof. Moreover, the sort of depth Baldwin locates in the proof does not fit neatly into the most natural way of interpreting Steiner's ideas about explanation. I also observe that Baldwin's discussion seems to conflate different general ideas about explanation that appear in the literature, but which have not been clearly distinguished.

The second part of the paper calls attention to the fact that mathematical explanations come in various forms, and notes some basic ways in which mathematical explanations can differ. It is observed that this in turn provides a reason for why various proposed theories of mathematical explanation tend to account better for some explanations than others. I then draw upon work on scientific explanation of Salmon to distinguish between two general ways of categorizing theories of mathematical explanation. The first may be broadly considered ontic accounts of explanation, with Steiner's account belonging in this category. The second group is broadly epistemological, and includes those accounts that take understanding to stem from some sort of organizational feature, as on Kitcher's unification view. I suggest that many mathematical explanations have this general character, regardless of whether they can be captured by Kitcher's particular theory. Finally, I return to Henkin's proof and suggest that at least some of its depth, and contribution to our understanding, is to be found in its organizational features.

Keywords: explanation, proof, completeness

Melisa Vivanco (The University of Texas - Rio Grande Valley). ***The Arbitrariness of Mathematical Proof Properties.***

Abstract. Mathematical explanations have long been a subject of both philosophical reflection and scientific inquiry. This paper addresses the challenging question of whether it is possible to establish clear conditions for what makes a mathematical proof explanatory. I argue that the problem of individuating proofs complicates any attempt to define or even characterize what constitutes an explanatory mathematical proof. As examples from diverse mathematical fields reveal, many properties used to distinguish proofs—and, by extension, their explanatory power—are often arbitrary, tied more to the representations of results than to the mathematical phenomena themselves.

The philosophy of mathematics has historically attempted to delineate what qualifies as a mathematical proof, from formalist approaches in ancient Greece to contemporary focuses on mathematical practice. These efforts have frequently resulted in seemingly more modest goals, such as characterizing specific types of proofs, particularly explanatory ones. Among the various properties proposed, symmetry (Lange, 2017) has emerged as a notable candidate for conferring explanatoriness. Nevertheless, the fact that multiple proofs can exist for the same theorem, and that mathematicians often disagree on which proofs are explanatory, suggests that attributes like symmetry may be arbitrary. This paper discusses proofs from algebra (D'Alembert's theorem), geometry (relations between trapezoid segments), and set theory (the cardinality of the power set). These examples illustrate that symmetry, while often ascribed explanatory power, is not unique to any particular area or methodology.

Ultimately, this work contends that the search for properties that universally endow proofs with explanatory power is unlikely to succeed. The arbitrary nature of such properties challenges the notion that they can be definitive in determining the explanatory value of a proof.

Selected References

Baker, A. (2005). Are There Genuine Mathematical Explanations of Physical Phenomena? Mind, 114(454), 223-238. Oxford University Press.

Bueno, O.; Vivanco, M. (2019). Resisting Non-causal Explanations. Analysis, 79(3), 550-559.

Lange, M. (2014). Aspects of Mathematical Explanation: Symmetry, Unity, and Salience. Philosophical Review 123 (4):485-531.

Lange, M. (2017). Because Without Cause. Oxford University Press..

Keywords: Mathematical explanation, Explanatory proof, Philosophy of mathematics, Mathematical practice, Symmetry

Jim Weatherall (University of California, Irvine) and Jesse Wolfson (University of California, Irvine). ***Correctness, Artificial Intelligence, and the Epistemic Value of Mathematical Proof.***

Abstract. We argue that it is neither necessary nor sufficient for a mathematical proof to have epistemic value that it be "correct", in the sense of formalizable in a formal proof system. There are two contexts for this claim. The first is the literature on the "Standard View" of mathematical proof (and of rigor) according to which "proper" proofs can all be formalized, and that it is something about this formalization (viz., its correctness as a formal derivation) that makes the proof adequate as justification for a mathematical claim. The second is a recent literature in mathematics exploring, and extolling, the role that future artificial intelligence systems could play in mathematical practice.

The argument that formal correctness is not sufficient is familiar: we observe than not all formal proofs are "interesting". The argument that it is not necessary has several parts. First, we argue that it cannot be necessary that a proof be formalizable in any fixed formal system. So, if formal correctness is necessary, then it must be that the proof is formalizable in *some system or other*. Then we argue that even this more general sense of necessity does not hold in the sociological sense, because mathematicians do not require the formalizability be demonstrated before they accept a proof as valid. Finally, we give examples of formally incorrect proofs that have had great epistemic value.

We then present a view on the relationship between mathematics and logic that clarifies the role of formal correctness in mathematics. The view we defend is one on which mathematical logic should be seen as a bit of applied mathematics: specifically, as a mathematical theory of mathematical practice. From this perspective, failures of formalizability can sometimes signal an error in a proof; but on the other hand, there can be (and have been) cases where a failure of formalizable is motivation to develop new formal systems that capture mathematically correct reasoning that has not previously be contemplated or theorized. More importantly, we argue, formal correctness cannot be what justifies mathematical arguments, any more than mathematical theories of physics can justify the motion of the planets.

Finally, we discuss the significance of these arguments for recent discussions about automated theorem provers and applications of AI to mathematics. We sketch a view of what we think AI mathematics could do, and also what the inherent limitations are given the relationship we describe between any kind of formal system and the human practice of mathematics.

Keywords: AI, informal proof, formal proof