The anabelian geometry of Grothendieck

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Chapman    26 May 2022
Grothendieck  1928-2014

• Mathematician ahead of his time

• Philosopher mathematician

• Isolated mathematician

• Outstanding contribution to Galois theory

Mathematical questions

• Analysis/Topology versus Algebra ?

• What is the precise gap between commutative and non-commutative mathematics ?
Évariste Galois  1811-1832

\[f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbb{Q}[X]\]

\(\text{Gal}(f)\)  finite group

\(f(X)\) solvable by radicals  \(\iff\) \(\text{Gal}(f)\) solvable

In general   \(K\) field   \(\overline{K} = K^{\text{alg}}\)

\[G_K = \text{Gal}(\overline{K}/K) \equiv \text{Aut}(\overline{K}/K)\]

Functor  \(\{\text{Fields}\} \xrightarrow{\text{Gal}} \{\text{Profinite Groups}\}\)

- What is the image of \(\text{Gal}\) ?

Fact  \(G_\mathbb{Q}\)  unknown

\(G_K\)  \(K\) infinite fin. gen.  mysterious!
Various approaches \( K/\mathbb{Q} \) finite

- Class Field Theory: explicit description of \( G_{ab}^b \)

- Iwasawa: understand \( G_{K}^{\text{metab}} \)

- Inverse Galois Problem: Hilbert, Shafarevich, 
  \( \ldots \)

- Galois representations: Weil, Shimura, Serre, Deligne, Faltings, Wiles, \( \ldots \)

- Langlands programme: \( L \)-functions, automorphic forms, representation
Grothendieck  SGA1  1960

$X$ connected scheme $\leadsto \pi_1(X, \ast)$

$X \rightarrow \text{Spec } K$  algebraic variety

$(* )$  $1 \rightarrow \pi_1(X)^{\text{geo}} \rightarrow \pi_1(X) \rightarrow G_K \rightarrow 1$

$(** )$  $\rho_X : G_K \rightarrow \text{Out } (\pi_1(X)^{\text{geo}})$

$X$ proper smooth curve  genus$(X) = g$

$$\Gamma_g = \prod_{i=1}^{g} \frac{\langle a_i, b_i \rangle_{i=1}^{g}}{[a_i, b_i]}$$

- char$(K) = 0$  $\pi_1(X)^{\text{geo}} \leadsto \Gamma_g$

- char$(K) = p > 0$  (Weil)

$$\pi_1(X)^{\text{geo}, (p')} \leadsto \Gamma_g^{\wedge, (p')}$$
Fundamental examples \( K = \mathbb{Q} \)

- \( X = E \) elliptic curve \( \pi_1(X)^{\text{geo}} \cong \hat{\mathbb{Z}}^2 \)

\[
\rho_X : G_{\mathbb{Q}} \to GL_2(\hat{\mathbb{Z}})
\]

- \( X = E \setminus \{0\} \) \( \pi_1(X)^{\text{geo}} \cong F_2 \)

\[
\rho_X : G_{\mathbb{Q}} \to \text{Out}(F_2)
\]

- \( X = \mathbb{P}^1 \setminus \{0, 1, \infty\} \) \( \pi_1(X)^{\text{geo}} \cong F_2 \)

\[
\rho_X : G_{\mathbb{Q}} \to \text{Out}(F_2)
\]

Grothendieck 1966: proof of Fermat?
Grothendieck anabelian conjectures (1980’s)

$K$ fin. gen. $\text{char}(K) = 0$

- **AN1** $L, F$ fin. gen. over $K$
  
  \[
  \text{Hom}_K(F, L) \to \text{Hom}_{G_K}(G_L, G_F)/ \sim
  \]
  
  is a bijection

- **AN2** $X, Y$ hyperbolic $K$-curves
  
  \[
  \text{Hom}_K(X, Y) \to \text{Hom}(\pi_1(X), \pi_1(Y))/ \sim
  \]
  
  is a bijection

- **Tate conjecture**: $A, B$ abelian varieties over $K$
  
  \[
  \text{Hom}_K(A, B) \otimes \hat{\mathbb{Z}} \to \text{Hom}_{G_K}(\pi_1(A)^{\text{geo}}, \pi_1(B)^{\text{geo}})
  \]
  
  is a bijection

- Arithmetic + topology $\implies$ rigid situation!
• **AN1** (isom form): Neukirch-Uchida (1970) Pop, Spiess (1990’s)

• **AN2** (isom form): Nakamura, Tamagawa, Mochizuki (1990’s)

• Mochizuki (1990’s): **AN1, AN2, K sub-$$p$$-adic field ($$p$$-adic Hodge theory)**

\[
\{\text{Fin. Gen. Fields }\} \xrightarrow{\text{Gal}} \{\text{Profinite Groups}\}
\]

\[
\{\text{Hyp. Curves }\} \xrightarrow{\pi_1} \{\text{Profinite Groups}\}
\]

• Images of "Gal" and "$$\pi_1$$" functors **mysterious**

**Aim** Improve this situation!
What is the meaning of **anabelian**?

Nakamura, Tamagawa, Mochizuki:

Here, the term **"anabelian algebraic variety"** means roughly "an algebraic variety whose geometry is controlled by its fundamental group, which is assumed to be 'far from abelian' ".

**False intuition !**

What is the **right anabelian geometry**?

**May 2017 **(*)
$m$-step solvable anabelian geometry

$G$ profinite group

- $\cdots \subseteq G(i+1) \subseteq G(i) \subseteq \cdots \subseteq G(1) \subseteq G(0) = G$
  
  $G(i+1) = [G(i), G(i)] \quad i \geq 0$

- $G^i \overset{\text{def}}{=} G/G(i) \quad i$-th step solvable quotient of $G$

$G^1 = G^{\text{ab}}, G^2 = G^{\text{metab}}, \ldots$

$j > i$:

\[
\begin{array}{cccccc}
1 & \longrightarrow & G(i) & \longrightarrow & G & \longrightarrow & G^i & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \text{id} & & \downarrow & & \\
1 & \longrightarrow & G[j,i] & \longrightarrow & G^j & \longrightarrow & G^i & \longrightarrow & 1
\end{array}
\]

- $K/\mathbb{Q}$ finite $\quad m \geq 1$

**Fact** Structure of $G^m_K$ can be approached via CFT (in principle)
Theorem A: \( L, F \) fin. gen. \( \dim(F) = \dim(L) = d \)

\[
\text{Isom}(F, L) \to \text{Isom}(\Gamma_L^m, \Gamma_F^m)/\sim
\]

is a bijection for all \( m \geq d^2 + 4d - 2 \).

Expected B: \( X, Y \) hyperbolic curves (fin. gen. fields)

\[
\text{Isom}(X, Y) \to \text{Isom}(\pi_1^m(X), \pi_1^m(Y))/\sim
\]

is a bijection for all \( m \geq 3 \).

\[
\pi_1(X)^m \to \Gamma_K^m
\]

Facts

- No need to know \( \Gamma_K \) in anabelian geometry!
- Theorem A **reconciles** anabelian geometry with CFT
- **Anabelian** world close to **abelian** world!
Mathematical philosophy of Grothendieck

What my experience of mathematical work has taught me again and again, is that the proof always springs from the insight, and not the other way round? and that the insight itself has its source, first and foremost, in a delicate and obstinate feeling of the relevant entities and concepts and their mutual relations. The guiding thread is the inner coherence of the image which gradually emerges from the mist, as well as its consonance with what is known or foreshadowed from other sources - and it guides all the more surely as the ”exigence” of coherence is stronger and more delicate.
How can we benefit more from Grothendieck Today?

- Embrace more his mathematical philosophy in our way of doing research in mathematics, and reconcile his philosophy with "practical mathematics".

- Embrace more his mathematical philosophy in our way of teaching mathematics.