Superoscillations and hyperfunctions
(Alain Yger, IMB, Université de Bordeaux, Talence, France)

Given $\omega \in \mathbb{C}$ with $|\omega| > 1$, the sequence $\{(\cos(z/N) + i\omega \sin(z/N))^N\}_{N\geq 1}$ converges towards $z \mapsto e^{i\omega z}$ in the algebra $A_1(\mathbb{C})$ of entire functions with growth controlled by $Ae^{B|z|}$ for some $A, B \geq 0$. This illustrates, when ω is additionally supposed to be real and functions are restricted to the real line, the concept of *super-oscillating sequence*. In the same vein, one may approximate the Cauchy kernel $z \mapsto 1/(z-x) = \lim_{N\to+\infty} 1/\big(N \sin((z-x)/N)\big)$ $(x \in \mathbb{R})$ in the algebra of homomorphic functions in $\mathbb{C} \setminus \mathbb{R}$ (the convergence being uniform with respect to $x \in \mathbb{R}$) by a sequence of entire functions such as $z \mapsto \sum_{j=0}^{N} \alpha_{N,j}(x) e^{i(1-2j/N)z}$, restricted to $\mathbb{C} \setminus \mathbb{R}$. This allows to carry the concept of *super-oscillating sequence* to the frame of *hyperfunctions on the real line*. I will thus recall to a necessarily non familiar audience such a concept and how it enlarges that of distribution, even that of so-called ultra-distribution (of the Gevrey type) on the real line. Then I will show that any hyperfunction on $\mathbb{R}$ with compact support or even just "tempered" (in a sense I will precise) can be approximated as an hyperfunction (in the sense of convergence "in terms of representatives") by a sequence of hyperfunctions represented by trigonometric polynomial functions with frequencies in $[-1, 1] \cap \mathbb{Q}$. I will then study evolution under Schrödinger type operators, such as in quantum physics or optics. I will thus decline within the frame of hyperfunctions in $U \subset \mathbb{R}^2$ the concept of *super-shift* and explain (quoting as an example the evolution of the family $\mathcal{F}$ of oscillating characters $x \mapsto e^{i\omega x} = \psi_\omega(t = 0, x)$, $\omega \in \mathbb{R}$, through the quantum harmonic oscillator $i\partial_t + (1/2)(\partial_x^2 - x^2)$) why such declination happens to be necessary in order that the super-shift phenomenom persists despite the apparition of singularities (even just locally integrable) for the evolved function $(t, x) \mapsto \psi_\omega(t, x)$, which is not the case if one keeps to the distribution setting. All results I will present are part of our recent joint work with Fabrizio Colombo, Irene Sabadini and Daniele Struppa.