2 TSQM has revealed new features and effects: Weak-Measurements

In other words, suppose first that the earlier measurement is that of $\hat{\sigma}_z$ and the later is that of $\hat{\sigma}_x$ (see fig. 6).

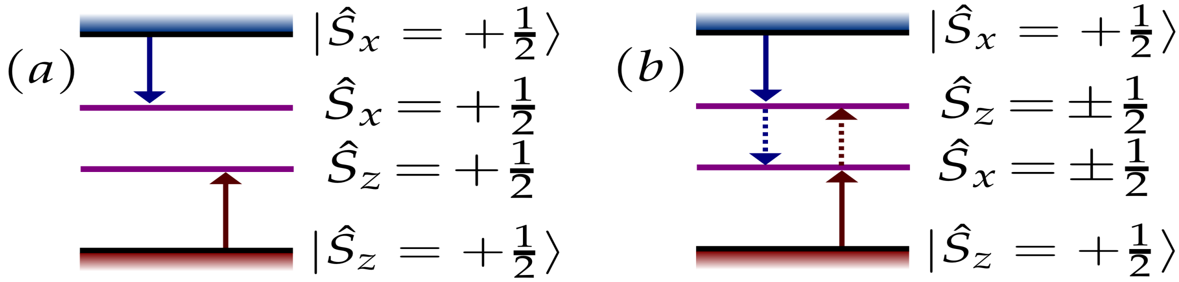


Figure 6: Measuring both the x and z spin components of a spin-1/2 particle in the interval between the pre- and post-selections of figure 4 yields different results depending on the time order of the two measurements. (a) If the first interim measurement measures the component set by the pre-selection, then both interim measurements are determined by the pre- and post-selections. (b) Reversing the order of the interim measurements destroys that certainty.

When measurements are performed in this order, it is obvious that at t_1 we obtain with certainty $\hat{\sigma}_z = +1$ - otherwise it would be inconsistent with the initial state - and the measurement at t_2 yields, also with certainty, $\hat{\sigma}_x = +1$ - otherwise it would be inconsistent with the final state. Suppose however that the order of the measurements is reversed, i.e. we measure $\hat{\sigma}_x$ at t_1 and $\hat{\sigma}_z$ at t_2 (fig 6.b). In this case each of these measurements can yield $+1$ as well as -1; all four possible combinations of results are consistent with the initial and final boundary conditions. ¿From the point of view of our two wave-functions formalism we understand this behavior in the following way. The information that $\hat{\sigma}_z = +1$, given by the initial condition and propagating forward in time cannot reach the time t_2 when we verify the value of $\hat{\sigma}_z$ because it is disrupted by the measurement of $\hat{\sigma}_x$ performed at t_1. Similarly, the information that $\hat{\sigma}_x = +1$, given by the final condition and propagating backward in time cannot reach the time t_1 when we verify the value of $\hat{\sigma}_x$ because it is disrupted by the measurement of $\hat{\sigma}_z$ at t_2.

Thus because the measurements of $\hat{\sigma}_z$ and $\hat{\sigma}_x$ do not yield with certainty $+1$ in both time orders, we shouldn't expect them to yield with certainty $+1$ when measured simultaneously (which is implied in measuring $\hat{\sigma}_{45} = 1/\sqrt{2}(\hat{\sigma}_x + \hat{\sigma}_z)$).

Now, once we understood the reason for which eq. 1.7 failed, the resolution of the problem is clear: If we would be able to somehow measure $\hat{\sigma}_x$ and $\hat{\sigma}_z$ such that they do not disturb each other, than we could measure them simultaneously and indeed obtain the paradoxical result $\hat{\sigma}_{45} = \sqrt{2}$.

Obviously as $[\hat{\sigma}_z, \hat{\sigma}_x] \neq 0$, the measurement of $\hat{\sigma}_z$ completely disturbs $\hat{\sigma}_x$ and vice-versa. However, at this point we note that it is actually possible to measure two non-commuting variables in such a way as not to disturb *completely* each other, if we *sacrifice precision*. This is the principle behind all macroscopic measurements. That's why classically (i.e. in the every-day semiclassical regime of quantum mechanics) we can measure, for example, *both* x and p and their measurements commute - in fact we measure them only approximatively, with combined precision far worse than $\Delta x \Delta p = \hbar$. So in general quantum mechanics offers the possibility of a trade off: one can gain non-disturbance by giving up precision. Let us now apply this idea to our case.

To be able to achieve a semi-classical regime we have to modify our problem. Instead of considering an ensemble of spin 1/2 particles we shall consider an ensemble of "particles", each "particle" being in fact formed by a large number N of spin 1/2 particles, such as a ferromagnet. As pre- and post-selected states we take now $|\hat{\sigma}_z = N\rangle$ and $|\hat{\sigma}_x = N\rangle$ respectively.

Now, the spin of a ferromagnet can be measured quite simply by measuring the magnetic field produced by it. For example, we can measure the magnetic field along any direction by the use of a piece of iron tied to a spring and oriented in an appropriate direction
As it is well-known, such measurements do commute with each other. What happens, of course, is that in such measurements the magnetic field is not measured to a very high precision. The flip of a few spins would hardly produce a significant modification of the spring elongation, far less than the uncertainty of the spring's length due to its quantum fluctuations. On the other hand, the interaction in between the piece of iron with each individual spin is very small, so that the spins are hardly disturbed, which is the reason why the measurements commute with each other. In principle it is quite possible to achieve a measurement of the magnetic field with error of, say, $\sqrt{N}$, while not disturbing more than $\sqrt{N}$ of the spins. But $\sqrt{N}$ is insignificant comparative to N when N is large, so that for large N we obtain very precise spin measurements which, at the same time,

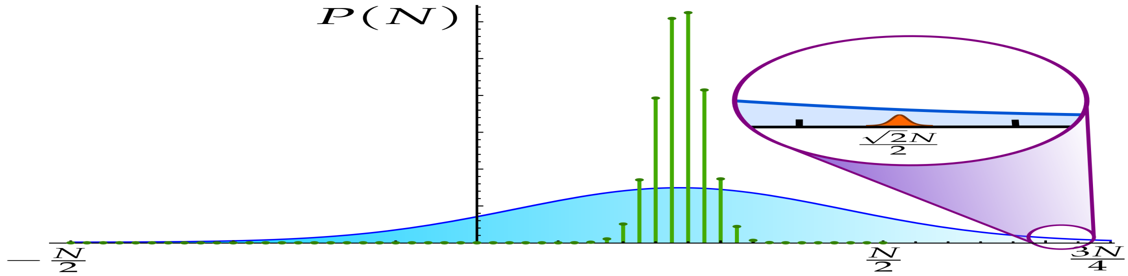


Figure 7: Probability distributions for different outcomes of the measurement of the spin component $\hat{\sigma}_{45}$ of a system of N spin-1/2 particle pre-selected in the state $|\hat{\sigma}_{\rm z}\rangle = +N/2$. Before any post-selection, the green histogram represents measurement outcomes for an ideal measurement. The blue curve represents the probabilities in an approximate measurement. After post-selection for very improbable state $|\hat{\sigma}_{\rm x}\rangle = +N/2$, only the red distribution way out in the tail survives.

produce very little disturbance. In this regime, since $\hat{\sigma}_{\rm z}$ and s_x effectively commute, they can also be measured separately. We therefore PREDICT that in this regime, the measurement of $\hat{\sigma}_{45}$ on our pre- and post-selected ensemble will yield, for almost each particle (ferromagnet)

$$\hat{\sigma}_{45} = \frac{\hat{\sigma}_{\rm x} + \hat{\sigma}_{\rm z}}{\sqrt{2}} = \frac{N+N}{\sqrt{2}} \pm O(\sqrt{N}) = \sqrt{2}N \pm O(\sqrt{N}), \tag{7}$$

i.e. a value completely outside the spectrum of the spin operator (which, in this case, extends only from $-N$ to N) !!!

To observe such a strange effect there is a price to pay - the probability of success in preparing the appropriate pre- and post-selection ensemble is very low. Indeed, we start with N spins polarized along the "up" z direction. As the "macroscopic" measurement of $\hat{\sigma}_{45}$ doesn't significantly disturb the spins, it follows that just before the post-selection measurement it is still the case that all the N spins of the ferromagnet are oriented "up" z. Each individual spin has therefore only a 1/2 chance to be found "up" x at the post-selection measurement. The total chance of finding all n spins "up" x is therefore 2^{-N}, i.e. an exponentially small number. Nonetheless, we are certain that when we were lucky enough and the post-selection succeeded, the intermediate measurement of $\hat{\sigma}_{45}$ yielded the paradoxical result of $\sqrt{2}N$.

The above prediction is far from trivial or intuitive from the point of view of the standard formalism of quantum mechanics. It is by no means a result which could be seen immediately from the beginning, by inspecting the state of the ensemble, as were the predictions about the vales of the x and z spin components. In order to verify it, we need to carefully model the "macroscopic" measurements described above - in the regime we considered we are far from the usual projection postulate, and the conditional probabilities formula eq. 1.7 doesn't apply anymore. We must then take into account that the measurements, while not completely disturbing the spins, are not completely precise and can yield errors. As we'll show bellow, the paradoxical result $\hat{\sigma}_{45} = \sqrt{2}N$ appears in the end through a mysterious conspiration of these measurement errors, which in turn,result from an intricate interference effect in the measuring device. Nevertheless, the two wave-functions formalism allows us to predict the result in a most simple and intuitive way. This is (one of) the major reasons why we propose it.

More formally, to explore the relationship between $|\Psi_{\rm in}\rangle$ and $|\Psi_{\rm fin}\rangle$ we must reduce the disturbance on the system during the intermediate time. If one doesn't perform absolutely precise (ideal) measurements but is willing to accept some finite accuracy, then one can bound the disturbance on the system. For example, according to Heisenberg's uncertainty relations, a precise measurement of position reduces the uncertainty in position to zero $\Delta x = 0$ but produces an infinite uncertainty in momentum $\Delta p = \infty$. On the other hand, if we measure the position only up to some finite precision $\Delta x = \Delta$ we can limit the disturbance of momentum to a finite amount $\Delta p \geq \hbar/\Delta$. Similarly, unlike finite rotations $\exp i\sigma_x\theta_x$ or $\exp i\sigma_y\theta_y$ around the x or y axis, the infinitesimal rotations $\exp i\epsilon\sigma_x$ and $\exp i\epsilon\sigma_x$ do in fact commute -their product being up to $O(\epsilon)^2$ corrections $1 + i\epsilon(\sigma_x + \sigma_y)$. This suggests that we should try to measure the spin σ_ξ by turning on a *weak* magnetic field B or turn it for a short time so that the precession angle around the ξ axis $\theta(\xi) \sim Bt$ will be small.

By replacing precise measurements with a bounded-measurement paradigm, we often find that paradoxical situations remain. Nevertheless, weak measurements produce surprising and often strange, but nevertheless consistent structures.

2.1 Quantum Measurements

In general, measurement of a quantum operator $\hat{A}$ is done by turning on and off an interaction between the quantum system of interest with wavefunction $|\Psi_{\rm in}\rangle$ and a measuring device: $H_{\rm int} = -\lambda(t)\hat{Q}_{\rm md}\hat{A}$ where $\hat{Q}_{\rm md}$ is an observable of the measuring-device (e.g. the position of the pointer) and $\lambda(t)$ is a coupling constant which determines the duration and strength of the measurement (the analog of the precession angle above). For an impulsive measurement we need the coupling to be strong and short and thus take $\lambda(t) \neq 0$ only for $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and set $\lambda = \int_{t_0-\varepsilon}^{t_0+\varepsilon} \lambda(t)dt$. We may then neglect the time evolution given by $H_{\rm s}$ and $H_{\rm md}$ in the complete Hamiltonian $H = H_{\rm s} +_{\rm md} + H_{\rm int}$. Using the Heisenberg equations-of-motion for the momentum $\hat{P}_{\rm md}$ of the measuring-device (conjugate to the position $\hat{Q}_{\rm md}$), we see that $\hat{P}_{\rm md}$ evolves according to $\frac{d\hat{P}_{\rm md}}{dt} = \lambda(t)\hat{A}$. Integrating this, we see that $P_{\rm md}(T) - P_{\rm md}(0) = \lambda\hat{A}$, where $P_{\rm md}(0)$ characterizes the initial state of the measuring-device and $P_{\rm md}(T)$ characterizes the final. To make a more precise determination of $\hat{A}$ requires that the shift in $P_{\rm md}$, i.e. $\delta P_{\rm md} = P_{\rm md}(T) - P_{\rm md}(0)$, be distinguishable from it's uncertainty, $\Delta P_{\rm md}$. This occurs, e.g., if $P_{\rm md}(0)$ and $P_{\rm md}(T)$ are more precisely defined and/or if λ is sufficiently large (see figure 8.a). However, under these conditions (e.g. if the measuring-device approaches a delta function in $P_{\rm md}$), then the disturbance or back-reaction on the system is increased due to a larger $H_{\rm int}$, the result of the larger $\Delta Q_{\rm md}$ ($\Delta Q_{\rm md} \geq \frac{1}{\Delta P_{\rm md}}$). When $\hat{A}$ is measured in this way, then any operator $\hat{O}$ ($[\hat{A}, \hat{O}] \neq 0$) is disturbed because it evolved according to $\frac{d}{dt}\hat{O} = i\lambda(t)[\hat{A}, \hat{O}]\hat{Q}_{\rm md}$, and since $\lambda\Delta Q_{\rm md}$ is not zero, $\hat{O}$ changes in an uncertain way proportional to $\lambda\Delta Q_{\rm md}$.

In the spin-1/2 example, the conditions for an ideal-measurement $\delta P^{\xi}_{\rm md} = \lambda\hat{\sigma}_\xi \gg \Delta P^{\xi}_{\rm md}$ will also necessitate $\Delta Q^{\xi}_{\rm md} \gg \frac{1}{\lambda\hat{\sigma}_\xi}$ which will thereby create a back-reaction causing a precession in the spin such that $\Delta\Theta \gg 1$ (i.e. more than one revolution), thereby destroying (i.e. making completely uncertain) the information that in the past we had $\hat{\sigma}_x = +1$, and in the future we will have $\hat{\sigma}_y = +1$.

In the Schroedinger picture, the time evolution operator for the complete system from $t = t_0 - \varepsilon$ to $t = t_0 + \varepsilon$ is $\exp\{-i\int_{t_0-\varepsilon}^{t_0+\varepsilon} H(t)dt\} = \exp\{-i\lambda\hat{Q}_{\rm md}\hat{A}\}$. This shifts the pointer, namely the momentum $P_{\rm md}$ (see figure 8.a) of the measuring device, by an amount δP_{md} proportional to $\lambda = \int \lambda(t)dt$ (the analog of the precession angle above) and the "value" of $\hat{A}$ in the state ψ. That "value" depends critically on the procedure of measurement.

Conventional Ideal or strong Von Neuman measurements have $\lambda >> 1$. More specifically the pointer or momentum shift δP^i_{md} is $\lambda.a_i$ when $\psi = |a_i>$ is any one of the eigenstates $|a_i>$ of the operator $\hat{A}$ with eigenvalue a_i. We demand that these shifts and differences thereof exceed the original quantum spread of the Pointer wave function $\Phi(P_{md}), \Delta P_{md}$.

Thus, after the experiment is done, the MD is at one of the distinct almost orthogonal states Φ^i, with P_{md} shifted by the different $\delta^i(P_{md})$. When the initial $|\Psi_{\rm in}\rangle$ is a superposition of A eigenstates $|\Psi_{\rm in}\rangle = \Sigma c_i|a_1\rangle$ the strong measurement yields the corresponding entangled superposition of particle and measuring device states:

$$\Sigma c_i|a_i\rangle\Phi^i P_{md}|a_i\rangle \tag{2.8}$$

The notion that such strongly shifted pointers behave classically and cannot interfere underlies the "collapse" where the above superposition becomes a statistical mixture or density matrix with weights $\rho_i = |c_i|^2$ - a feature which cannot fully be explained within the framework of linear evolution.

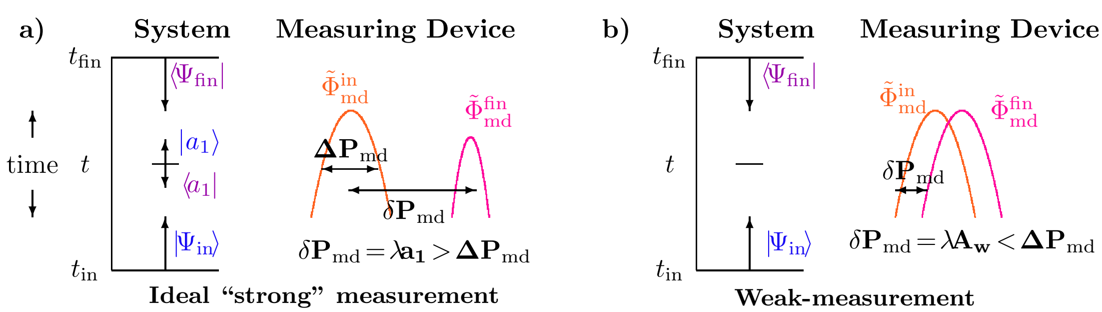
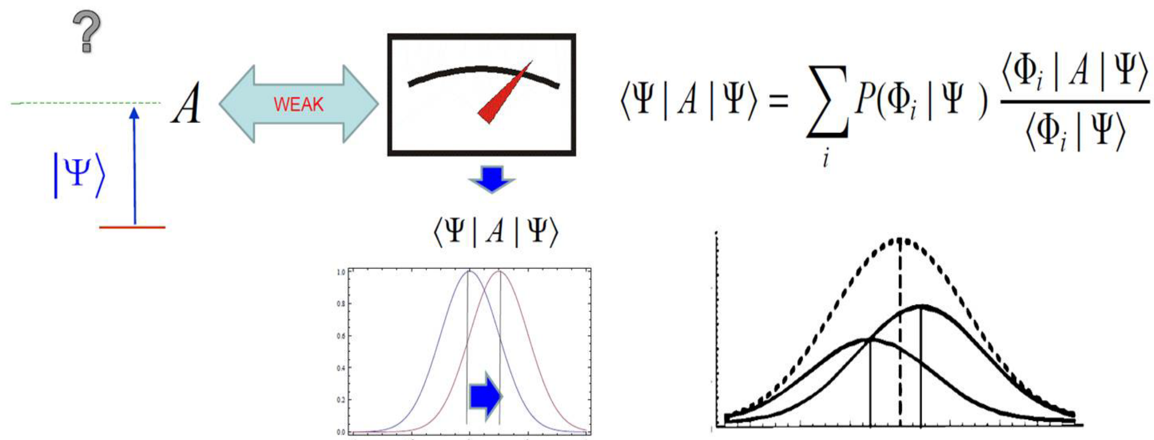


Figure 8: a) with an ideal or "strong" measurement at t (characterized e.g. by $\delta P_{\rm md} = \lambda a_1 \gg \Delta P_{\rm md}$), then ABL gives the probability to obtain a collapse onto eigenstate a_1 by propagating $\langle\Psi_{\rm fin}|$ backwards in time from $t_{\rm fin}$ to t and $|\Psi_{\rm in}\rangle$ forwards in time from $t_{\rm in}$ to t; in addition, the collapse caused by ideal-measurement at t creates a new boundary condition $|a_1\rangle\langle a_1|$ at time $t \in [t_{\rm in}, t_{\rm fin}]$; b) if a weak-measurement is performed at t (characterized e.g. by $\delta P_{\rm md} = \lambda A_w \ll \Delta P_{\rm md}$), then the outcome of the weak-measurement, the weak-value, can be calculated by propagating the state $\langle\Psi_{\rm fin}|$ backwards in time from $t_{\rm fin}$ to t and the state $|\Psi_{\rm in}\rangle$ forwards in time from $t_{\rm in}$ to t; the weak-measurement does not cause a collapse and thus no new boundary condition is created at time t.

2.1.1 Weakening the interaction between system and measuring device



We however focus on the opposite case of weak measurements. The interaction $H_{\rm int} = -\lambda(t)\hat{Q}_{\rm md}\hat{A}$ is weakened by minimizing $\lambda\Delta Q_{\rm md}$ so that the pointer shift for any of the $|a_i\rangle$ states and for superpositions thereof is smaller than the uncertainty $\Delta(P_{md})$ in the pointer (momentum) of the measuring device which we take as a Gaussian wave function of width $\Delta(P_{md}) = 1$ (see Fig) Thus the hardly shifted MD cannot instigate a collapse.

We may then set $e^{-i\lambda\hat{Q}_{\rm md}\hat{A}} \approx 1 - i\lambda\hat{Q}_{\rm md}\hat{A}$ and use a theorem[1]:

$$\hat{A}|\Psi\rangle = \langle\hat{A}\rangle|\Psi\rangle + \Delta A|\Psi_\perp\rangle\,, \tag{2.9}$$

to show that the initial quantum state evolves under the weak measurement according to:

$$e^{-i\lambda\hat{Q}_{\rm md}\hat{A}}|\Psi_{\rm in}\rangle = (1 - i\lambda\hat{Q}_{\rm md}\hat{A})|\Psi_{\rm in}\rangle = (1 - i\lambda\hat{Q}_{\rm md}\langle\hat{A}\rangle)|\Psi_{\rm in}\rangle - i\lambda\hat{Q}_{\rm md}\Delta\hat{A}|\Psi_{\rm in\perp}\rangle \tag{2.10}$$

The norm of this state $\parallel (1 - i\lambda\hat{Q}_{\rm md}\hat{A})|\Psi_{\rm in}\rangle \parallel^2 = 1 + \lambda^2\hat{Q}^2_{\rm md}\langle\hat{A}^2\rangle$. Hence, the probability to leave $|\Psi_{\rm in}\rangle$ unchanged after the weakened measurement approaches unity:

$$\frac{1 + \lambda^2\hat{Q}^2_{\rm md}\langle\hat{A}\rangle^2}{1 + \lambda^2\hat{Q}^2_{\rm md}\langle\hat{A}^2\rangle} \longrightarrow 1 \quad (\lambda \to 0) \tag{2.11}$$

while the probability to disturb the state (i.e. to obtain $|\Psi_{in\perp}\rangle$) is:

$$\frac{\lambda^2\hat{Q}^2_{\rm md}\Delta\hat{A}^2}{1 + \lambda^2\hat{Q}^2_{\rm md}\langle\hat{A}^2\rangle} \longrightarrow 0 \quad (\lambda \to 0) \tag{2.12}$$

2.1.2 Information gain without disturbance: safety in numbers

The key observation is that the probability to disturb the state decreases as $O(\lambda^2)$, but the shift of the measuring-device grows linearly $O(\lambda)$, so $\delta P_{\rm md} = \lambda a_i$. For a sufficiently weak interaction (e.g. $\lambda \ll 1$), the probability for disturbing the state can be made arbitrarily small, while the measurement still yields information. For any given weak measurement, the shift in the measuring-device is much smaller than its uncertainty $\delta P_{\rm md} \ll \Delta P_{\rm md}$ (figure 8.b) so we really cannot use this information.

However, if a large ($N \geq \frac{N'}{\lambda}$) number of particles is used, then the shift of all the measuring-devices ($\delta P^{tot}_{\rm md} \approx \lambda\langle\hat{A}\rangle\frac{N'}{\lambda} = N'\langle\hat{A}\rangle$) becomes distinguishable because of repeated integrations, while the collapse probability still goes to zero.

Classical deterministic measurements of $\hat{A}$, i.e. of it's average in the state of interest, also do not disturb that state and are of the form discussed. A topical example is provided by LIGO . This lazer interferometric device measures the tiny change in distance $\delta(d)$ 10^{-14} cm between two mirrors induced by the passage of a gravity wave . the resulting shift in the interference pattern is less than a 10 billionth of a fringe is observable by using the many coherent identical photons provided by the laser to obtain a net effect on the large classical wave.

[1] where $\langle\hat{A}\rangle = \langle\Psi|\hat{A}|\Psi\rangle$, $|\Psi\rangle$ is any vector in Hilbert space, $\Delta A^2 = \langle\Psi|(\hat{A} - \langle\hat{A}\rangle)^2|\Psi\rangle$, and $|\Psi_\perp\rangle$ is a state such that $\langle\Psi|\Psi_\perp\rangle = 0$.

2.1.3 Adding a post-selection to the weakened interaction: Weak-Values and Weak-Measurements

Having established that weak measurements allow information to be gained without disturbance- it is fruitful to inquire whether this type of measurement reveals new features (in the context of TSQM). With weak-measurements (which involve adding a post-selection to this ordinary -but weakened- von Neumann measurement), the measuring-device registers a new value, the weak-value. As an indication of this, we insert a complete set of states $\{|\Psi_{\rm fin}\rangle_j\}$ into the outcome of the weak interaction of §2.1.1 (i.e. the expectation value $\langle\hat{A}\rangle$):

$$\langle\hat{A}\rangle = \langle\Psi_{\rm in}|\left[\sum_j |\Psi_{\rm fin}\rangle_j\langle\Psi_{\rm fin}|_j\right]\hat{A}|\Psi_{\rm in}\rangle = \sum_j |\langle\Psi_{\rm fin}|_j\Psi_{\rm in}\rangle|^2\ \frac{\langle\Psi_{\rm fin}|_j\ \hat{A}\ |\Psi_{\rm in}\rangle}{\langle\Psi_{\rm fin}|_j\Psi_{\rm in}\rangle} \tag{2.13}$$

If we interpret the states $|\Psi_{\rm fin}\rangle_j$ as the outcomes of a final ideal-measurement on the system (i.e. a post-selection) then performing a weak-measurement (e.g. with $\lambda\Delta Q_{\rm md}\to 0$) during the intermediate time $t\in[t_{\rm in},t_{\rm fin}]$, provides the coefficients for $|\langle\Psi_{\rm fin}|_j\Psi_{\rm in}\rangle|^2$ which gives the probabilities $Pr(j)$ for obtaining a pre-selection of $\langle\Psi_{\rm in}|$ and a post-selection of $|\Psi_{\rm fin}\rangle_j$. The intermediate weak-measurement does not disturb these states and the quantity $A_{\rm w}(j)\equiv\frac{\langle\Psi_{\rm fin}|_j\hat{A}|\Psi_{\rm in}\rangle}{\langle\Psi_{\rm fin}|_j\Psi_{\rm in}\rangle}$ is the weak-value of $\hat{A}$ given a particular final post-selection $\langle\Psi_{\rm fin}|_j$. Thus, from the definition $\langle\hat{A}\rangle=\sum_j Pr(j)\,A_{\rm w}(j)$, one can think of $\langle\hat{A}\rangle$ for the whole ensemble as being constructed out of sub-ensembles of pre-and-post-selected-states in which the weak-value is multiplied by a probability for a post-selected-state.

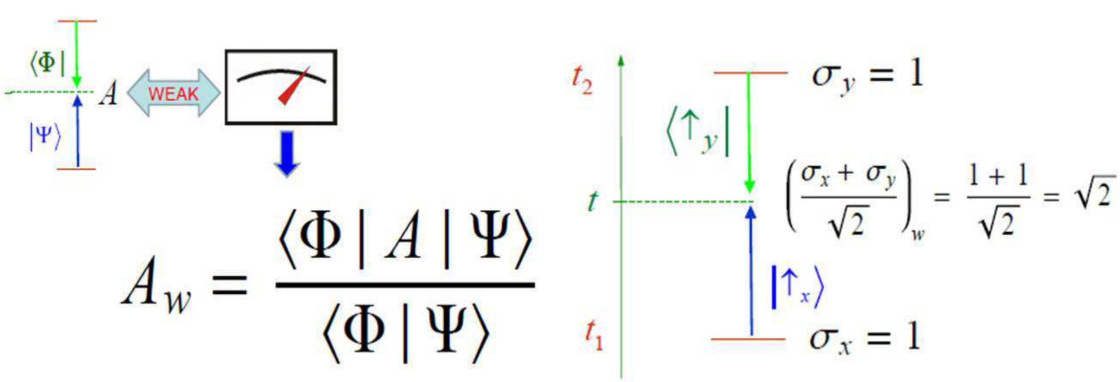
The weak-value arises naturally from a weakened measurement with post-selection: taking $\lambda << 1$, the final state of measuring-device in the momentum representation becomes:

$$\begin{aligned}
\langle P_{\rm md}|\langle\Psi_{\rm fin}|e^{-i\lambda\hat{Q}_{\rm md}\hat{A}}|\Psi_{\rm in}\rangle|\Phi^{\rm MD}_{\rm in}\rangle &\approx \langle P_{\rm md}|\langle\Psi_{\rm fin}|1+i\lambda\hat{Q}_{\rm md}\hat{A}|\Psi_{\rm in}\rangle|\Phi^{\rm MD}_{\rm in}\rangle\\
&\approx \langle P_{\rm md}|\langle\Psi_{\rm fin}|\ \Psi_{\rm in}\rangle\{1+i\lambda\hat{Q}\frac{\langle\Psi_{\rm fin}|\hat{A}|\Psi_{\rm in}\rangle}{\langle\Psi_{\rm fin}|\Psi_{\rm in}\rangle}\}|\Phi^{\rm MD}_{\rm in}\rangle\\
&\approx \langle\Psi_{\rm fin}|\Psi_{\rm in}\rangle\langle P_{\rm md}|e^{-i\lambda\hat{Q}A_{\rm w}}|\Phi^{\rm MD}_{\rm in}\rangle\\
&\to \langle\Psi_{\rm fin}|\Psi_{\rm in}\rangle\exp\left\{-(P_{\rm md}-\lambda\,A_{\rm w})^2\right\} \qquad (2.14)\\
where\ \ A_{\rm w} &= \frac{\langle\Psi_{\rm fin}|\hat{A}|\Psi_{\rm in}\rangle}{\langle\Psi_{\rm fin}|\Psi_{\rm in}\rangle}
\end{aligned}$$

The final state of the measuring-device is shifted by a very unusual quantity, the weak-value, $A_{\rm w}$, which is not in general an eigenvalue of $\hat{A}$

We have used such limited disturbance measurements to explore many paradoxes. A number of experiments have been performed to test the predictions made by weak-measurements and results have proven to be in very good agreement with theoretical predictions Since eigenvalues or expectation values can be **derived** from weak-values we believe that the weak-value is indeed of fundamental importance in QM. The weak-value is the relevant quantity for all generalized weak interactions with an environment, not just measurement interactions. The only requirement being that the 2-vectors, i.e. the pre-and-post-selection, are not significantly disturbed by the environment.

2.1.4 How the weak-value of a spin-1/2 can be 100



Let's finally return to the spin 1/2 system. There **is** a sense in which both $Prob_{\rm ABL}(\hat{\sigma}_x=+1)=1$ *and* $Prob_{\rm ABL}(\hat{\sigma}_y=+1)=1$ are simultaneously relevant because measurement of one does *not* disturb the other.

Since measurement of $\hat{\sigma}_\xi$ also can be understood as a simultaneous measurement of $\hat{\sigma}_x$ *and* $\hat{\sigma}_y$, with limited-disturbance measurements, we can simultaneously use both $\hat{\sigma}_x = +1$ and $\hat{\sigma}_y = +1$ to obtain

$$(\hat{\sigma}_{\xi=45^\circ})_{\mathrm{w}} = \frac{\langle \uparrow_y | \frac{\hat{\sigma}_y + \hat{\sigma}_x}{\sqrt{2}} | \uparrow_x \rangle}{\langle \uparrow_y | \uparrow_x \rangle} = \frac{\{\langle \uparrow_y | \hat{\sigma}_y\} + \{\hat{\sigma}_x | \uparrow_x \rangle\}}{\sqrt{2}\langle \uparrow_y | \uparrow_x \rangle} = \frac{\langle \uparrow_y | 1 + 1 | \uparrow_x \rangle}{\sqrt{2}\langle \uparrow_y | \uparrow_x \rangle} = \sqrt{2} \tag{2.15}$$

This was confirmed experimentally for an analogous observable, the polarization. The weak-value is $\sqrt{2}$ times bigger than the allowed eigenvalue.