Systemic Portfolio Diversification *

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Abstract

We study the implications of fire-sale externalities on balance sheet composition. Banks select their asset holdings to minimize expected execution costs triggered by the need to comply with regulatory leverage requirements. Our analysis highlights the fundamental trade-off between asset diversification at the level of each individual bank and systemic diversification. While sacrificing diversification benefits to reduce portfolio commonality increases the bank's idiosyncratic probability of liquidation, it also lowers the endogenous probability of a costly widespread sell-off. We show that leverage heterogeneity is socially beneficial because it amplifies banks' incentives in achieving systemic diversification. The socially optimal systemic diversification can be attained through a tax on banks' balance sheet concentration on illiquid assets.

Key words: systemic diversification, leverage, fire-sale externalities
JEL Classification: G01, G21, G38

1 Introduction

The classical paradigm in financial investment prescribes asset diversification as a means to minimize risk. Standard pre-crisis policies argued for the unlimited benefits of diversification, with little emphasis on balancing those against the downside risks of contagion. However, the global 2007-2009 financial crisis highlighted potential vulnerabilities resulting from balance sheet interconnectedness of financial institutions: in a crisis, investors exposed to the same shocked asset may be forced to simultaneously liquidate their positions in this asset. The liquidation of an asset carried out simultaneously by many financial institutions exacerbates losses for all investors involved in the sell-off.

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Prior literature has analyzed the vast amount of banks deleveraging that occurred during the crisis, and studied the feedback between tightening liquidity and falling asset prices (e.g., Brunnermeier and Pedersen (2008), Khandani and Lo (2011), Manconi et al. (2012)). Fire-sale spillovers due to asset commonality among banks have been recognized as a major source of systemic risk (e.g., Allen and Carletti (2013), Billio et al. (2012)). Concerns about fire-sale externalities led, for example, to the initiation of asset purchase programs under TARP by the U.S. Treasury and to the emergency cash bailout of Bear Stearns by J.P.Morgan Chase and the New York Federal Reserve in March, 2008.[1] All this indicates the importance of balancing asset diversification (optimal in isolation) with the diversification of liquidation risk *across banks*.

A financial firm may mitigate the idiosyncratic risk of each asset by holding a diversified portfolio. This reduces the portfolio's variance, and therefore the firm's individual probability of asset liquidation. At the system level, instead, "systemic" diversification, i.e., the reduction of portfolio overlaps across different institutions, lowers the likelihood of concurrent asset liquidation and, therefore, of costly widespread sell-offs. Figure 1 illustrates this trade-off, and in particular how portfolio commonality may lead to a higher probability of simultaneous asset liquidation.

We consider a financial system consisting of banking institutions subject to a regulatory leverage constraint: if after an initial market shock the asset value of a bank falls and its resulting leverage exceeds a given threshold, then the bank is required to liquidate assets to return to its regulatory requirement. Asset liquidation is costly, and imposes a downward pressure on prices proportional to the quantity that is being liquidated. A bank is then exposed to cross-agent externalities if its portfolio significantly overlaps with the portfolios of other banks facing similar constraints. Each bank chooses its asset holdings ex-ante, i.e., before the market shock is realized, accounting for potential vulnerability to fire-sale spillovers. Reducing portfolio overlapping lowers the negative externalities resulting from cross holdings.

The proposed model highlights the mechanism through which systemic risk affects the banks' portfolio choice, and quantifies the externalities imposed by the banks on the system. We argue that systemic risk from fire-sale spillovers should play an important role on the banks' balance sheet decisions: a portfolio that is optimal for an agent in isolation may be far from optimal if cross-agent externalities are accounted for. Our analysis shows that even though banks reduce portfolio commonality to mitigate the risk of fire-sale spillovers, they do not reduce it enough relative to the social optimum. This result is a consequence of the fact that each bank only accounts for the costs that other banks impose on it, but disregards the externalities imposed on the rest of the system through its liquidation actions. Our study allows to assess the efficacy of welfare-enhancing policies. Any regulatory intervention based on the banks' current balance sheet allocations is subject to the Lucas critique, as banks may adapt to the new regulatory environment in unexpected and, potentially, socially damaging ways. Hence, to anticipate the feedback effects following regulatory

[1] The provision of liquidity by the Federal Reserve was taken to avoid a potential resale of nearly U.S. $210 billion of Bear Stearns' assets. The Chairman of the Fed, Ben Bernanke, defended the bail-in by stating that Bear Stearns' bankruptcy would have affected the economy, causing a "chaotic unwinding" of investments across the U.S. markets and a further devaluation of other securities across the banking system.

intervention, policies should account for the banks' optimal response, i.e., their equilibrium asset allocations. Monetary policy tools such as asset purchase programs run by government or regulatory bodies may have the unintended consequence of incentivizing banks to hold excessively correlated financial exposures. For example, Acharya et al. (2010) argue that providing unconditional liquidity support to banks decreases their incentives to hold a liquid portfolio. We show that imposing a tax on the banks' balance sheet interconnectedness may align the private optimum with the socially optimal asset allocation. The externality that a bank imposes on the system is increasing in the size of its balance sheet, the liquidity of its asset holdings, its leverage ratio, and the illiquidity-weighted portfolio overlap with other banks. This externality, and the corresponding Pigovian tax, is related to the systemicness of a bank, as defined in Greenwood et al. (2015): the tax is a weighted average of a modified version of the bank's systemicness over different asset shocks.[2]

Our model predicts that a higher heterogeneity in the financial system reduces the expected aggregate liquidation costs. Even if each bank were to ignore fire-sale spillovers, it would still select the portfolio based on its leverage ratio because the latter determines the incentives of holding liquid assets. Therefore, banks with different leverage hold different portfolios. When banks account for fire-sale spillovers, the diversity in banks' portfolios becomes even higher because each bank runs away from the externalities imposed by the others. From a policy perspective, these findings suggest that mergers of banks may have unintended consequences, and increase the fragility of the system. This is because consolidation reduces the level of heterogeneity in the system and, as a result, the possibility of diversifying fire-sale risk across banks. While a single large bank may optimally choose its asset allocation, it may not be able to diversify its liquidation risk. By contrast, in a system of two heterogeneous banks, each bank can adjust its portfolio to lower the likelihood of joint asset liquidation.

A growing financial literature analyzes the aggregate vulnerability of the empirically observed banking system to fire-sale risk (e.g Greenwood et al. (2015),Capponi and Larsson (2015), and Duarte and Eisenbach (2018)). Unlike these studies, we do not take banks' portfolios as exogenously given. We frame the decision making problem of banks' portfolio selection as a game in which each bank minimizes the expected value of its own asset liquidation costs. Each bank's allocation decision affects the likelihood and magnitude of forced asset liquidations and the bank's contribution to systemic risk. We show that this game may be casted as a potential game, and therefore a Nash equilibrium can always be guaranteed to exist under mild conditions on the distribution of the initial market shock. There are multiple economic forces that affect the Nash equilibrium of the game. First, portfolio diversification reduces each bank's likelihood of forced liquidation; second, highly leveraged banks have a stronger incentive to hold liquid assets as they are more vulnerable to fire sales; third, banks seek to reduce portfolio commonality to limit fire-sale spillover costs. Because of the intricate patterns of interactions of these three forces, the game may admit multiple

[2] In the U.S., banks file the form FR Y-9C every quarter with the Federal Reserve, a report that collects their consolidated balance sheet data. This information allows a regulator to monitor common exposures in the financial system, infer –after accounting for size and leverage ratio– each bank's contribution to systemic risk, and impose a tax that makes the bank internalize such a contribution.

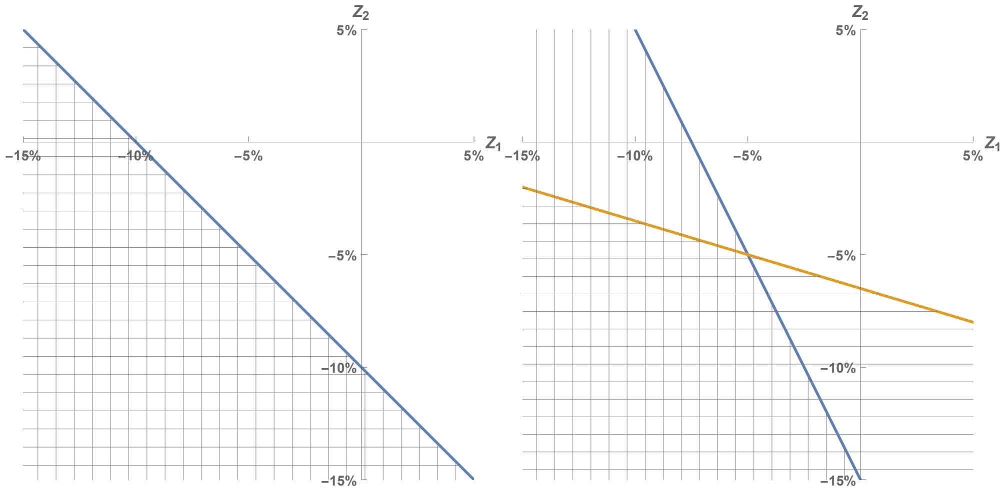

Figure 1: An economy consisting of two banks and two assets. Z_1 and Z_2 are the returns of assets 1 and 2, respectively. Solid lines represent the portfolio composition of each bank. A bank is forced to liquidate assets if its portfolio return causes leverage constraint to be breached. The horizontal lines identify the asset returns for which bank 1 liquidates assets, and the vertical lines the asset returns for which bank 2 liquidates assets. Left panel: both banks hold the same perfectly diversified portfolio, hence –depending on the shock on the assets– either they are both forced to liquidate assets, or neither bank is. Right panel: banks hold different portfolios, as a consequence, the region in which both banks liquidate simultaneously is smaller.

equilibria.[3] We show that a financial system that is sufficiently homogeneous in both the banks' characteristics (size and leverage) and the assets' liquidity levels admits a unique equilibrium. In such a setting, the unique equilibrium is that the more leveraged bank adjusts its position towards the more liquid asset.

Literature Review

Existing literature has analyzed the implications on asset pricing and financial stability resulting from banks' leverage management. Adrian and Shin (2010) are the first to provide empirical evidence that banks react to asset price changes by actively managing their balance sheets. Greenwood et al. (2015) introduce a model to explain the propagation of shocks in a system of leverage-targeting banks with common asset holdings. They focus on the first-order effects of fire-sale losses caused by spillovers, and measure the contribution of each bank to the fragility of the system: a highly connected bank, i.e., a bank that holds assets to which many other banks have large exposures, is a source of vulnerability for the system. Capponi and Larsson (2015) generalize their analysis and introduce the systemicness matrix to show that higher-order effects of fire-sale externalities

[3]Consider a system with two assets and two banks, and assume that the more leveraged bank is significantly smaller. In one equilibrium, the more leveraged bank holds a larger position in the liquid asset than the bigger bank, because its incentive to hold the liquid asset is stronger. In another equilibrium, the lowly leveraged bank, which is dominant in the system because of its size, adjusts its portfolio towards the liquid asset. The smaller bank, whose incentive to run away from the externality imposed by the bigger bank now dominates the incentive to hold the liquid asset, increases its position in the illiquid asset.

can be substantial during periods of financial distress. Duarte and Eisenbach (2018) empirically study the historical vulnerability to fire-sale spillovers of American banks, including the periods before and after the financial crisis. "Illiquidity concentration", i.e., the concentration of illiquid assets among large and levered banks, is shown to have increased significantly up to early 2007. This measure demonstrates the importance of balance sheet linkages in the propagation of market shocks, and corroborates our claim that large banks should account for the portfolio composition of other systemically important banks. Cont and Wagalath (2013) show that in the event of large market shocks, distressed market participants jointly liquidate assets. These sell-offs result in spikes of correlation, reducing the benefits of asset diversification when it is most needed. In all these studies, banks are assumed to be endowed with a given portfolio and liquidate assets in response to an exogenous shock. Our study considers instead banks that are strategic in the choice of their portfolios.

Our work is also related to existing literature on counterparty risk networks. Acemoglu et al. (2015*a*) study the resilience to shocks of different network architectures. They conclude that a completely interconnected system, i.e., in which all institutions completely diversify their counterparty credit risk, may increase the fragility of the system if a large shock hits the network. A similar behavior is observed in the network of portfolio holdings, where two institutions share a link if their portfolios overlap: in an interconnected network multiple agents hold similar portfolios and, after a large market shock, they may all be forced to simultaneously sell assets, exacerbating the costs for all agents participating in the sell-off. While Acemoglu et al. (2015*a*) analyze an ex-post scenario where shocks have already hit the balance sheet structures of banks in the network, we consider an ex-ante scenario where the shock is yet to occur. Our network of portfolio holdings is endogenously determined by the equilibrium choice of banks' asset holdings.

Farboodi (2017) and Acemoglu et al. (2015*b*) consider endogenous intermediation and highlight the inefficiencies arising from overexposure to counterparty risk by banks which make risky investment. Our work shows that, even in the absence of direct credit linkages, banks are exposed to excessive systemic risk because in equilibrium they hold portfolios that are too similar.

Other related studies include Acharya and Yorulmazer (2007), who show that banks may find it optimal to invest in highly correlated assets in anticipation of a bailout triggered by many simultaneous failures; and Farhi and Tirole (2012), who support Acharya and Yorulmazer (2007)'s findings by showing that safety nets can provide perverse incentives and induce correlated behavior that increases systemic risk. Unlike these studies, we focus on aggregate vulnerability rather than defaults, and do not model default resolution policies such as bailouts.

Our work is also related to the study by Wagner (2011) on the trade-off between diversity and diversification in financial exposures. While his model also explores the implications of joint asset liquidation on the agents' portfolio allocations, our focus on a finite number of systemically important financial institutions requires the design of a different framework and leads to considerably different conclusions. In our model, it is heterogeneity that drives banks to reduce their portfolio overlaps. In a homogeneous setup, banks would hold the same portfolio. In contrast, in the model by Wagner (2011) even if agents are identical their portfolios are not. This is because he considers a continuum of agents, each of negligible size, and therefore each agent is not subject to any liquidation cost if it sells assets while all others do not. Furthermore, because in our model agents are large, it is possible to quantify the externalities that each bank imposes on the rest of the system and analyze policy remedies.[4]

Our study on the efficacy of a tax on systemic risk is related to the work by Acharya et al. (2017). They construct an aggregate indicator for the occurrence of a systemic crisis, which is exogenously specified in terms of total assets and capital of the banks in the system. Each bank is charged a tax in the amount equal to the share of expected aggregate loss it generates during this crisis scenario. Different from their top-down approach, we infer the tax directly from the banks' balance sheet information: the tax amount is equal to the endogenous cost that each bank imposes on the rest of the system due to simultaneous asset liquidation, rather than being determined via an exogenously defined systemic event.

The rest of the paper is organized as follows. We introduce the model primitives and the economic assumptions in Section 2. We describe the game theoretical model of strategic banks' holdings in Section 3. We solve for the Nash equilibrium and discuss its properties in Section 4. We study the social planner problem and discuss policy implications in Section 5. We discuss operational challenges in Section 6. Section 7 concludes the paper. Technical proofs are deferred to the Appendix.

2 Model Setup

We consider a two-period economy consisting of K assets and N banks. Let d_i be the initial debt of bank i, and e_i its initial equity. Denote by $w_i := d_i + e_i$ the total initial asset value of bank i. The leverage ratio of bank i is

$$\lambda_i := \frac{d_i}{e_i},$$

which may be equivalently rewritten as $d_i = \frac{\lambda_i}{1+\lambda_i} w_i$. Each bank is required to maintain its leverage below a regulatory threshold λ_M, i.e., $\lambda_i \leq \lambda_M$ for every i.

At date 1, each bank chooses its asset allocations. Denote by $\pi_{i,k}$ the weight of asset k in bank i's portfolio. Portfolio weights are positive and satisfy the relation $\sum_{k=1}^{K} \pi_{i,k} = 1$ for $i = 1, \cdots, N$.

At date 2, each asset k is subject to a return shock Z_k. Hence, the return of bank i's portfolio is $R_i = \pi_i^T Z$, where $\pi_i := (\pi_{i,k})_{1 \leq k \leq K}$ is the vector of bank i's weights and $Z := (Z_k)_{1 \leq k \leq K}$ is the vector of shocks. Ex-post, the leverage ratio of bank i is

$$\lambda_i^{post} = \frac{d_i}{w_i(1 + R_i) - d_i}.$$

Assumption 2.1. *After a shock, banks that violate the leverage requirement liquidate the minimum*

[4] Other studies have considered different settings to investigate the dangers and social costs of asset diversification, e.g., Shaffer (1994), Acharya (2009), Ibragimov et al. (2011).

amount of assets needed to adjust their leverage to the regulatory threshold.

If $\lambda_i^{post} \geq \lambda_M$, then bank i needs to sell assets and use the cash proceeds from the sale x_i to repay its debt. To comply with the leverage requirement, x_i needs to satisfy $\frac{d_i - x_i}{w_i(1+R_i)-d_i} = \lambda_M$. The focus of the paper is on large liquidity shocks, and it is well known that raising equity during distressed market conditions is prohibitively costly. Similarly, selling assets in a depressed market environment is difficult. Nevertheless, empirical evidence provided by Adrian and Shin (2008) – see the scatter plot in Figure 6 therein– indicates that firms manage leverage primarily through adjustments in the size of debt (e.g. through asset disposals), leaving equity unchanged, rather than through direct changes in equity.[5]

We remark that Greenwood et al. (2015) assume that banks target their leverage, i.e., that they immediately sell assets to return to their initial leverage ratio. In our model it is only the assets required by the bank to meet the leverage constraint that are liquidated at discounted prices. A bank that intends to restore its initial leverage ratio may do so on a longer time scale to incur lower execution costs.

Assumption 2.2. *Banks liquidate assets proportionally to their initial allocation.*

As in Greenwood et al. (2015) and Duarte and Eisenbach (2018), we assume that if bank i needs to raise a total amount of cash x, then it liquidates $\pi_{i,k}x$ for each asset k. This assumption may be interpreted as a stationarity condition on the composition of the banks' portfolio: in a hypothetical multi-period model, the proportional liquidation strategy would yield a terminal portfolio that is close to the initial portfolio, and therefore still resilient to subsequent market shocks. We remark here that there is no agreement in the empirical literature on the liquidation strategy adopted by financial firms when they liquidate assets. On the one hand, selling liquid assets first reduces the cost of fire sales. On the other hand, holding liquid assets carry an option value because of the prospectus that markets may become more illiquid in the future (see Ang et al. (2014)). Because of the lack of conclusive evidence on what force dominates in this trade-off, we consider the case of proportional liquidation. [6]

Assumption 2.3. *Asset liquidation is costly.*

If a large return shock causes the bank's leverage to exceed the regulatory threshold, the bank needs to immediately readjust its positions to comply with regulatory requirements. If assets are liquidated on a very short notice, then the bank may need to sell them at discounted prices

[5]If banks use a combination of equity issuance and asset liquidation to comply with regulatory requirements, the size of fire-sale externalities would be lower but our qualitative conclusions would remain unaltered.

[6]Greenwood et al. (2015) and Duarte and Eisenbach (2018) also consider –as an alternative to the proportional liquidation strategy– a pecking order of liquidation where banks first sell off their most liquid assets. They show that in a calibrated model of fire-sale spillovers this strategy reduces the magnitude of fire-sale losses. In the same context of leverage targeting, Capponi and Larsson (2015) show analytically that fire-sale externalities are smaller if banks first sell liquid and then illiquid assets. In a two-period game-theoretical model, forcing banks to follow a pecking order strategy may lead to counterintuitive results: all banks would simultaneously first sell the most liquid asset, making its liquidation costly. Some banks may therefore prefer not to hold any share of the most liquid asset only as an artificial consequence of the pecking order constraint.

relative to their fundamental values, i.e., a fire sale would occur. The initial price of each asset k is normalized to one dollar. If the aggregate amount of asset k that banks liquidate is q_k, the execution price *per share* of the asset is

$$p_k^{post} := 1 + Z_k - \gamma_k q_k,$$

where $\gamma_k > 0$ is the price impact parameter of asset k. The limiting case $\gamma_k \downarrow 0$ corresponds to the case of a perfectly liquid asset.

Our model abstracts from the underlying source of market illiquidity, and captures the knocked down effect of sales on prices in reduced form through the parameter γ. The price impact function can be viewed as a representation of outside investors with limited capital and other investment opportunities. This form of price impact function captures the mechanics of typical theoretical models of fire sales, as explained next. Suppose outside investors with a fixed dollar amount of outside wealth were to step in and provide liquidity to the banking sector during a fire sale. These investors would then face a trade-off between the returns from investment in outside projects with gains from investing in fire sold assets; see also Shleifer and Vishny (2011) for a related discussion.

The total amount of shares of asset k that banks liquidate is

$$\sum_{j=1}^{N} \pi_{j,k} x_j \chi_{\{\lambda_j^{post} > \lambda_M\}},$$

where $\chi_{\{\lambda_j^{post} > \lambda_M\}}$ is the indicator function of the event $\{\lambda_j^{post} > \lambda_M\}$. Hence, the liquidation cost per share of asset k is $\gamma_k \sum_{j=1}^{N} \pi_{j,k} x_j \chi_{\{\lambda_j^{post} > \lambda_M\}}$. Let $Diag[\gamma]$ be the diagonal matrix with entries γ_k on the diagonal. Then the total liquidation cost incurred by bank i at time 2 is

$$cost_i(\pi_i, \pi_{-i}, Z) := \underbrace{x_i \chi_{\{\lambda_i^{post} > \lambda_M\}} \pi_i^T}_{\text{assets liquidated by bank } i} \quad \mathrm{Diag}[\gamma] \quad \underbrace{\sum_{j=1}^{N} \pi_j x_j \chi_{\{\lambda_j^{post} > \lambda_M\}}}_{\text{total quantities traded}},$$

where $\pi_{-i} := (\pi_1, \cdots, \pi_{i-1}, \pi_{i+1}, \cdots, \pi_N)$.

We assume that assets' returns are identically distributed. Maximizing the banks' expected portfolio return is therefore equivalent to minimizing their expected liquidation costs. Even though banks are risk-neutral, the leverage constraint introduces a cost that depends on each bank's tail risk, which can be interpreted as a form of induced risk preference.

At date 1, bank i chooses the portfolio weights that minimize the expected value of its total liquidation cost incurred at date 2. We outline the technical assumptions A.2–A.5 in the Appendix. We refer to them whenever they are required in our results.

3 Equilibrium Asset Allocations

The banks' portfolio allocations are described by a game theoretical model as follows:

i) The N banks are the *players*;

ii) The set $X := \{x \in [0,1]^K \ : \ \sum_{k=1}^K x_k = 1\}$ of admissible portfolio weights is the space of *strategies*;

iii) Each bank minimizes an objective function given by the expected total liquidation cost, i.e.,

$$\mathrm{EC}_i(\pi_i, \pi_{-i}) := E[\mathrm{cost}_i(\pi_i, \pi_{-i}, Z)].$$

A Nash equilibrium is a set of banks' asset allocation decisions $\pi := (\pi_1, \cdots, \pi_N)$ such that no bank has any incentive to unilaterally deviate from it, i.e., $\mathrm{EC}_i(\pi_i, \pi_{-i}) \le \mathrm{EC}_i(\tilde{\pi}_i, \pi_{-i})$, for any i and strategy $\tilde{\pi}_i$ of bank i. Bank i's objective function may be rewritten as $S_i(\pi_i) + \sum_{j \neq i} M_{i,j}(\pi_i, \pi_j)$, where

$$\begin{aligned} S_i(\pi_i) &:= E\left[\pi_i^T \mathrm{Diag}[\gamma] \pi_i x_i^2 \chi_{\{\lambda_i^{post} > \lambda_M\}}\right], \\ M_{i,j}(\pi_i, \pi_j) &:= E\left[\pi_i^T \mathrm{Diag}[\gamma] \pi_j x_i x_j \chi_{\{\lambda_i^{post} > \lambda_M\} \cap \{\lambda_j^{post} > \lambda_M\}}\right]. \end{aligned}$$

S_i is the idiosyncratic component of the expected liquidation cost incurred by bank i. Such a cost is due to the price dislocation caused by bank i's asset sales, and would persist even in the absence of other banks in the system. $M_{i,j}$ is the systemic component of the liquidation cost, i.e., the extra cost incurred by bank i due to the presence of bank j in the system. This precisely captures the externality that bank j imposes on bank i due to their overlapping portfolios.

Fix the asset holdings π_{-i} of all other banks except i. Bank i's optimization problem is equivalent to choosing the portfolio weight vector π_i that minimizes

$$P(\pi) := \sum_{m=1}^N \left(S_m(\pi_m) + \sum_{j<m} M_{m,j}(\pi_m, \pi_j) \right).$$

Because each bank minimizes the same objective function, the problem can be formulated in terms of a *potential game*, where $P(\pi)$ is the potential function.

Proposition 3.1. *The game specified by (i)-(iii) is a potential game. Moreover, if Z is a continuous random variable with values in $\mathbb{R}^K$, the game admits a Nash equilibrium.*

Even though a Nash equilibrium exists, its uniqueness cannot be always guaranteed. This is due to the complex trade-offs faced by each bank in the system. On the one hand, diversification reduces the likelihood of asset liquidation. On the other hand, concentration on liquid assets and avoidance of portfolio overlapping with other banks reduces the realized liquidation costs. This multiplicity of economic forces that drive allocation decisions imply that the bank's optimization problem is in

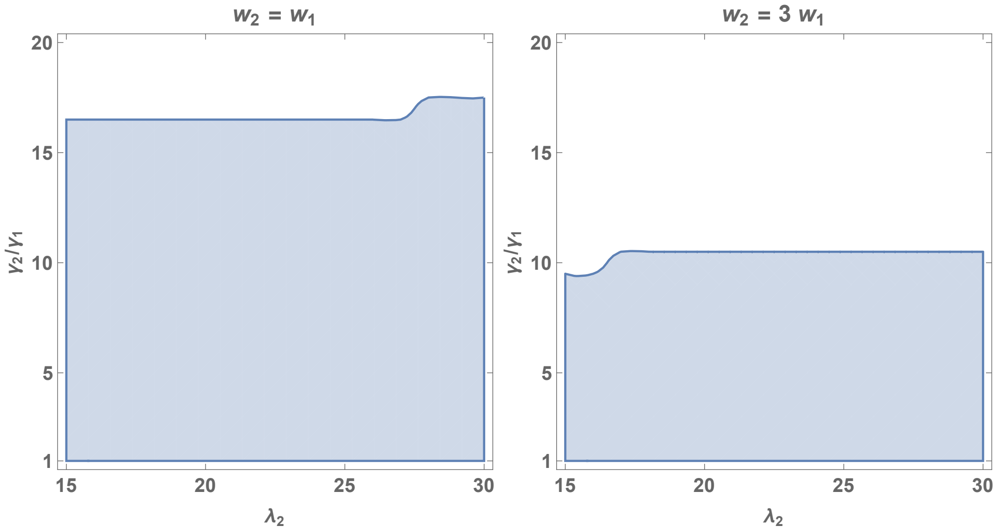

Figure 2: The region of parameters in which the function $P(\pi)$ is convex for $N = 2$, $K = 2$. The leverage threshold λ_M is set to 30 and λ_1 is set to 15.

general non-convex. Hence, the game may admit multiple equilibria. If the system is sufficiently homogeneous, then the incentives to diversify and to hold a liquid portfolio are sufficiently aligned, and the game admits a unique equilibrium (see Theorem 3.2). In Figure 2 we illustrate that the equilibrium is unique even in a moderately heterogeneous system.

Theorem 3.2. *Let $N = 2$, $K = 2$. Under Assumption A.2, for any $\lambda, w, \gamma > 0$ there exist $\lambda_* < \lambda < \lambda^*$, $w_* < w < w^*$, $\gamma_* < \gamma < \gamma^*$ such that if $\lambda_i \in (\lambda_*, \lambda^*)$, $w_i \in (w_*, w^*)$ for $i = 1, 2$ and $\gamma_k \in (\gamma_*.\gamma^*)$ for $k = 1, 2$, then there exists a unique Nash equilibrium.*

4 The Nash Equilibrium of Banks' Portfolio Holdings

The optimal portfolio allocation of a bank in isolation, i.e., for $N = 1$, provides a benchmark for analyzing the impact of the system on the equilibrium allocation. If either all assets have the same liquidity or all banks are equally leveraged, then the presence of other institutions in the system does not affect the bank's portfolio holdings. However, in the case of a heterogeneous financial system, each bank seeks to run away from the systemic externalities by reducing its portfolio commonality with other banks.

4.1 Single Bank Benchmark

In the absence of systemic externalities, the bank's portfolio allocation decision is driven by two main forces: the likelihood of breaching the leverage constraint and the total realized costs from the bank's liquidation strategy. If all assets are equally liquid, the individual minimization of these two criteria yields the same outcome: complete *diversification* is optimal (see Proposition 4.1 for the formal statement). First, the portfolio's variance –hence the probability of violating the leverage constraint– is minimized when the portfolio is fully diversified. Second, the marginal cost of asset liquidation is increasing in the quantity that is sold, therefore liquidating smaller positions in multiple assets results in a lower cost than liquidating a large position in a single asset.

Proposition 4.1. *Let $N = 1$ and $\gamma_k = \gamma$ for each k. Under Assumption A.2, the optimal allocation is $\pi^S_{1,k} = \frac{1}{K}$ for all k.*

If assets have different liquidity levels, the two forces described above drive the bank's allocation decisions in opposite directions. The probability of violating the leverage constraint does not depend on the liquidity of the assets, but only on the distribution of the assets' returns. On the one hand, a portfolio with equal asset weights minimizes the likelihood of forced liquidation because returns are identically distributed. On the other hand, to reduce the total costs in the event of a forced sale the bank should allocate a larger portion of its wealth to the most liquid assets. More precisely, in a market consisting of two assets with liquidity parameters $\gamma_1 < \gamma_2$, the optimal liquidation policy is attained when the marginal cost from the sale of each asset is identical. In other terms, if a bank has to liquidate assets, it is optimal to sell a proportion x of the first asset and $1 - x$ of the second asset, where x satisfies the indifference condition $1 + \gamma_1 x = 1 + \gamma_2(1 - x)$, i.e., $x = \frac{\gamma_2}{\gamma_1+\gamma_2}$. Because of the trade-off between these two forces, the optimal portfolio weight $\pi^S_{1,1}$ lies in the interval $(\frac{1}{2}, \frac{\gamma_2}{\gamma_1+\gamma_2})$.

Proposition 4.2. *Let $N = 1$, $K = 2$, and $0 < \gamma_1 < \gamma_2$. Assume that $S_1(\pi_1)$ is convex. Under Assumption A.2, it holds that $\pi^S_{1,1} \in (\frac{1}{2}, \frac{\gamma_2}{\gamma_1+\gamma_2})$, where $(\pi^S_{1,1}, 1 - \pi^S_{1,1})$ minimizes the function S_1 on X. Furthermore, $\pi^S_{1,1}(\lambda)$ is an increasing function of λ.*

Proposition 4.2 also states that banks with different leverage ratios weigh these two forces differently, and therefore hold different optimal portfolios. A highly leveraged bank is more likely to breach the leverage constraint regardless of its portfolio allocations, and should therefore aim at reducing the costs of its asset liquidation strategy. Vice versa, a bank with a low leverage ratio is less concerned about its realized liquidation costs, and constructs a portfolio that is more diversified but less liquid.

4.2 Homogeneous Economy

Consider an economy that is homogeneous either across assets –all assets have the same liquidity– or across banks –all banks are equally leveraged–. Then, banks hold identical portfolios in equilibrium. Furthermore, as formalized in the next Proposition, this portfolio coincides with that of the single bank benchmark in which each bank does not account for the liquidation actions of other banks.

Proposition 4.3. *Under Assumptions A.2 and A.3:*

(1) If $\gamma_k = \gamma$ for each k, then $\pi^*_{i,k} = \frac{1}{K}$ is the unique Nash equilibrium.

(2) Let π^S be the vector of optimal weights determined in Proposition 4.2 when the leverage ratio is λ. If $\lambda_i = \lambda$ for each i, then $\pi^*_i = \pi^S$ is the unique Nash equilibrium.

While the presence of other institutions holding the same portfolio π^* results in higher expected liquidation costs, it does not alter the portfolio holdings of each individual bank. To see this, consider two equally leveraged banks in a market with two assets. The optimal portfolio π^* that each bank holds if it were the only institution in the system is such that the expected marginal liquidation costs for each asset are identical. If this were not the case, the bank would invest more in the asset with lower marginal cost to reduce the total expected costs. However, in a system with two banks, each bank also accounts for the externalities imposed by the other bank. These externalities can be decomposed across assets: a higher portfolio weight in an asset implies a larger externality resulting from the liquidation of that asset. If both banks hold the same portfolio π^*, the additional cost that each bank imposes on the other due to portfolio commonality is the same for all assets. In particular, the marginal expected liquidation costs for each asset after accounting for the additional cost imposed by the other bank are also identical, because these additional externalities are the same across assets.

4.3 Heterogeneous Economy

In this section, we consider an economy in which there is heterogeneity both with respect to assets' illiquidity and banks' leverage. Then the systemic externalities arising from joint asset liquidation affect the banks' optimal portfolio allocations. As argued in Section 4.1, a bank whose leverage is closer to the regulatory threshold values liquid assets more than a bank with lower leverage. In a market consisting of two heterogeneous assets, the bank with lower leverage holds a more diversified portfolio and allocates a higher proportion of wealth to the more illiquid asset relative to the more leveraged bank. Hence, the presence of the bank with smaller leverage contributes to increase the costs of the illiquid asset held by the highly leveraged bank, which in turn readjusts its portfolio to hold an even larger position in the liquid asset. Analogously, the less leveraged bank shifts its portfolio towards the less liquid asset. Both banks adjust their positions further and the prevailing Nash equilibrium is the aggregate outcome of this process. In each iterative step of this procedure, each bank runs away from the externalities imposed by the other bank in the system.

Theorem 4.4 states that banks reduce portfolio overlapping, and therefore cross-bank externalities, when they account for the presence of other banks in the system.

Theorem 4.4. *Let $N = 2$, $K = 2$. Assume $\gamma_1 < \gamma_2$ and $\lambda_1 > \lambda_2$. Under Assumptions A.2, A.3, and A.4, $|\pi^*_{1,1} - \pi^*_{2,1}| > |\pi^S_{1,1} - \pi^S_{2,1}|$, where $\pi^S_{i,1}$ is bank i's optimal asset 1 allocation in the single bank benchmark.*

Taken together, Proposition 4.3 and Theorem 4.4 show that it is the heterogeneity in the financial system that gives banks incentives to reduce their common exposures. We provide a graphical illustration of this phenomenon in Figure 3. When assets have the same liquidity, i.e., $\gamma_1 = \gamma_2$, all banks hold the same perfectly diversified portfolio. If the assets have different liquidity, then each bank's optimal portfolio allocation would be different even in its corresponding single bank benchmark. Cross-bank externalities increase diversity in banks' asset holdings, because banks seek

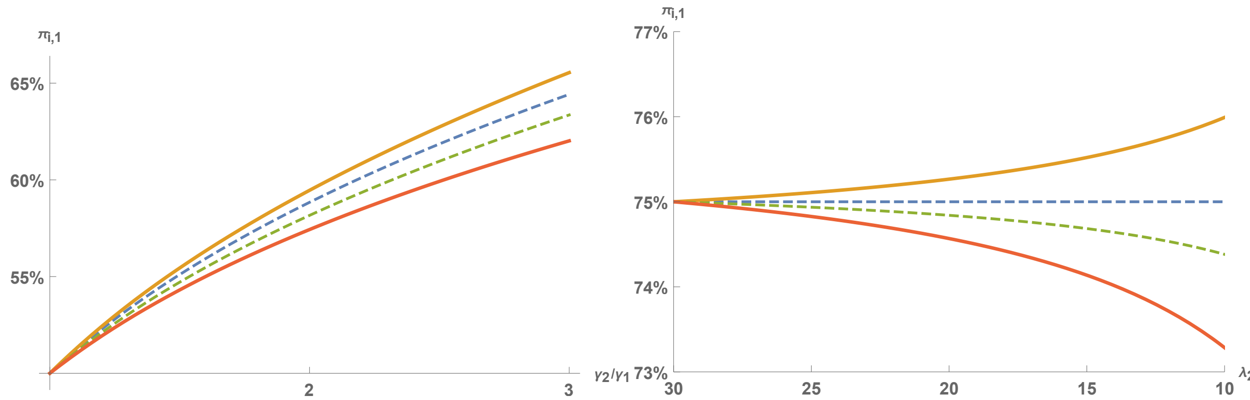

Figure 3: Banks' portfolio allocations in asset 1 in an economy consisting of two assets and two banks. We increase heterogeneity across assets (left panel) and across banks (right panel). Solid lines represent the allocations in the two-bank economy (orange for bank 1, red for bank 2), dashed lines the allocations of each bank in the single-bank benchmark (blue for bank 1, green for bank 2). We fix $\lambda_M = \lambda_1 = 30$. In the left panel, we choose $\lambda_2 = 5$. In the right panel, we choose $\gamma_2/\gamma_1 = 6$.

to reduce their portfolio commonality. A similar mechanism arises when considering heterogeneity in banks' leverages: assuming assets have different liquidity levels, banks hold identical portfolios when they are equally leveraged. Vice versa, if banks have different leverage ratios they reduce their common exposures significantly compared to the portfolio commonality implied by the optimal allocations in their corresponding single bank benchmark (see the right panel in Figure 3).

Even though banks may not have accounted for systemic externalities created by portfolio commonality prior to the global 2007-2009 financial crisis, empirical evidence appears to suggest that they have started to account for the risk of fire-sale spillovers in more recent years. Duarte and Eisenbach (2018) define a measure of portfolio overlap on illiquid assets by large leveraged banks, called "illiquidity concentration". This measure has increased steadily until 2007 and started to drop in 2013. The recent decrease in illiquidity concentration signifies either a higher awareness of portfolio contagion or the efficacy of new regulatory measures.

4.4 A System with Multiple Equilibria

In general, the uniqueness of a Nash equilibrium cannot be guaranteed even in a financial system consisting of only two banks and two assets. This is because the cost function is not necessarily convex in the portfolio weights. To understand why this is the case, consider two banks of significantly different size: a highly leveraged small bank and a lowly leveraged large bank. Because the externalities imposed by the small bank on the large bank are small relative to the size of the latter bank, the large bank's portfolio allocations are close to the ones it would choose if it were the only bank in the system. The small bank balances the following economic forces. First, the small bank would like to reduce the likelihood of breaching the leverage constraint, and hence it aims at holding a fully diversified portfolio. Second, the highly leveraged small bank has a stronger incentive to hold the liquid asset because this lowers the cost of asset liquidation. Third, to minimize its portfolio

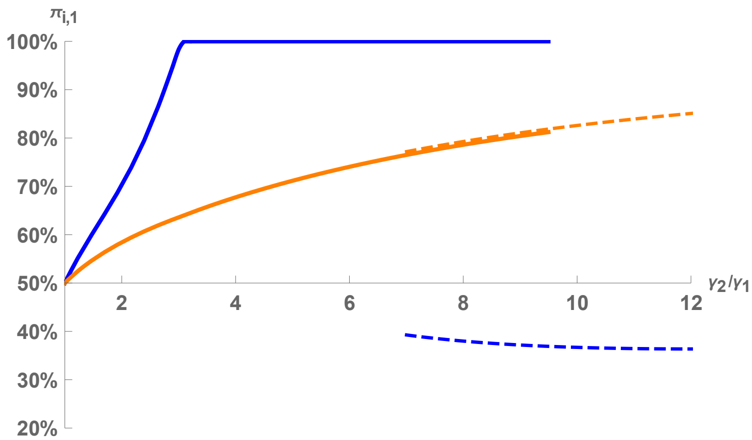

Figure 4: Banks' portfolio allocations in asset 1 for a market of two banks and two assets as a function of the assets' heterogeneity. Solid lines represent the allocations in one equilibrium (orange for bank 1, blue for bank 2), dashed lines the allocations of each bank in another equilibrium. We fix $\lambda_M = 30$, $\lambda_1 = 10$, $\lambda_2 = 25$, $w_1 = 50w_2$. There exist two prevailing equilibria in the γ_2/γ_1 interval [7, 9.5].

overlap with the large bank, the small bank may significantly increase its position in either asset. Hence, there exists a region of the parameter space in which two equilibria are possible: the small bank may run away from the externalities imposed by the large bank by either holding a much higher position in the liquid asset compared to the one it would hold in the absence of the large bank, or shifting its portfolio towards the illiquid asset. We illustrate these two potential outcomes in Figure 4. If the difference in liquidity of the two assets increases, then the large bank also holds a significant position in the liquid asset resulting in further portfolio overlap. If the expected liquidation costs due to the presence of the large bank are prohibitively high, it is preferable for the small bank to hold more shares of the illiquid asset and therefore reduce the portfolio overlap with the large bank. In the example of Figure 4, for values of relative asset illiquidity γ_2/γ_1 that belong to the interval [7, 9.5], two Nash equilibria exist.

5 Social Welfare and Policy Implications

An ample literature has discussed the systemic risk implications of fire sales (see, for example, Shleifer and Vishny (2011) and Schwarcz (2008)). Persistent price-drops may lead investors to lose confidence and withdraw funds from institutions, undermining financial intermediation and weakening the wider economy. If the assets held by the banks are bonds used to finance long term projects of state, local governments, or private companies, then premature liquidation of these assets results in a loss of economic productivity because it hampers the ability of running projects till maturity. We assume that the objective of the social planner is to minimize the expected aggregate liquidation costs in the system, i.e., the objective function $TC(\pi) = \sum_{i=1}^{N} EC_i(\pi)$.

The following result shows that in a heterogeneous economy the banks' asset allocations obtained in equilibrium are not socially optimal. Even if banks account for the presence of other banks and reduce portfolio overlapping, their holdings still exhibit excessive asset commonality relative to the social optimum.

Theorem 5.1. *Let $K = 2$ and $N = 2$. Under the assumptions of Theorem 4.4 and Assumption A.5, $|\pi_1^{SP} - \pi_2^{SP}| > |\pi_1^* - \pi_2^*|$, where π_i^{SP} is the bank i's asset 1 allocation that minimizes the social planner's objective function.*

Theorem 5.1 states that banks choose to hold excessively overlapping exposures compared to the social optimum. This is because each bank does not internalize the externalities it imposes on all other banks, but only accounts for the externalities imposed by other banks on itself when it makes its allocation decision. By contrast, in a homogeneous economy the social optimum is aligned with the banks' portfolio holdings obtained in equilibrium. In other words, if banks are equally leveraged, the equilibrium prescribed by Proposition 4.3 is socially optimal. In this equilibrium all banks hold the same portfolio, i.e., there is complete portfolio overlap. However, this does not imply that a homogeneous economy is socially preferable to a heterogeneous one. As the following Proposition shows, the opposite result holds, i.e., aggregate liquidation costs are maximized in a homogeneous economy.

Proposition 5.2. *Assume $K = 2$ and $N = 2$. Let w be the aggregate asset value and d the aggregate debt in the system. Assume that the banks' individual asset values are $w_1 = w_2 = \frac{w}{2}$, and that the debt levels d_1 and d_2 are such that $\lambda_1 := \frac{d_1}{w_1 - d_1} \leq \lambda_M$, $\lambda_2 := \frac{d_2}{w_2 - d_2} \leq \lambda_M$ where $d_1 + d_2 = d$. Define $TC^*(d_1)$ to be the total expected liquidation costs in equilibrium when the debt of bank 1 is d_1 (and therefore the debt of bank 2 is $d - d_1$). Then $d_1 = \frac{d}{2}$ is a local maximum of $TC^*(d_1)$.*

Theorem 4.4 showed that it is the heterogeneity in the banking system to drive banks towards a reduced portfolio overlap. Lower portfolio commonality in turn reduces the likelihood and severity of liquidity crises. Therefore, aggregate costs are higher in a homogeneous economy, where all banks hold the same portfolio, as stated in Proposition 5.2. See also Figure 5 for a plot of the total expected liquidation costs as a function of the heterogeneity in the system.

The probability of joint asset liquidation by all banks is determined endogeneously in equilibrium. In fact, the probability of a fire sale depends on the portfolio holdings of each bank in the system. As portfolio commonality increases, so does the likelihood of a fire sale. Consider for example an economy with two identical banks and two assets. Portfolio commonality is minimized when bank 1 only holds asset 1 and bank 2 only holds asset 2. Hence, a fire sale occurs when a large shock hits simultaneously both assets. If only one asset is hit by a market shock, one bank will be forced into asset liquidation, but not the other. By contrast, if both banks hold the same diversified portfolio then either both banks would be forced to jointly liquidate assets or neither of them would. This leads to a fundamental trade-off between asset diversification on the individual firm level and systemic portfolio diversification: while reducing portfolio commonality by holding

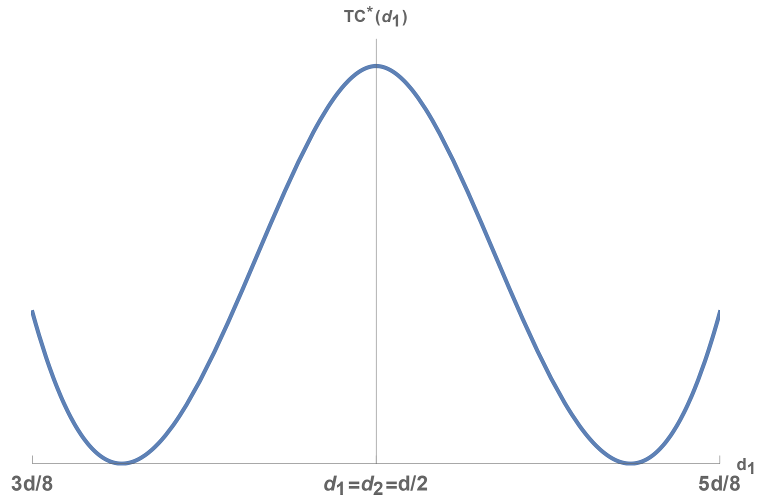

Figure 5: Total expected liquidation costs for different levels of leverage heterogeneity. If $d_1 = d_2$, then the system is homogeneous.

less diversified portfolios may increase the probability of liquidation in isolation, it also makes simultaneous liquidation less likely. The endogenous probability of a fire sale is therefore lower in an economy where, in equilibrium, banks reduce the portfolio overlap. In particular, as shown in Proposition 5.2, heterogeneity in the financial system is socially beneficial because it drives banks towards holding diverse portfolios.

From a policy making standpoint, Proposition 5.2 implies that bank mergers have an adverse impact on financial stability. A homogeneous system behaves as a single large bank, and can thus be interpreted as the outcome of bank consolidation. A single bank cannot diversify its liquidation risk, as it is either affected by the liquidation event or not. If instead the system is heterogeneous, banks manage their assets to account for systemic externalities, i.e., adjust their portfolios to reduce the likelihood of simultaneous sell-offs. Hence, systemic portfolio diversification occurs. A consolidated banking system decreases the available options for diversifying fire-sale risk across banks. Hence, the total quantity that the system is required to liquidate cannot be as optimally controlled as in a system with multiple smaller banks.

Next, we discuss how the imposition of a tax on the interconnectedness of the banking system may align the private banks' incentives with the social optimum. The next Proposition provides an explicit formula for such a tax, through which each bank fully internalizes the externalities imposed on the rest of the system.

Proposition 5.3. *Under Assumptions A.3 and A.5, if each bank i is charged a tax in the amount equal to $T_i(\pi) := \sum_{j \neq i} M_{i,j}(\pi)$, then the private equilibrium allocation is equal to the social planner's optimum.*

The tax amount $T_i(\pi)$ is equal to the sum of externalities $M_{i,j}(\pi)$, $j \neq i$, that bank i imposes on every other bank in the system. This externality is increasing in the size of the bank's balance sheet, bank's leverage ratio and the concentration of the bank's holdings on illiquid assets. This tax changes the ex-ante banks' incentives, aligning their equilibrium asset allocations with the social optimum. In practice, a tax on portfolio overlapping may be combined with the initiation of an asset purchase program in the event of a liquidity crisis. The tax would not only incentivize banks to reduce their common exposures, and hence the likelihood of asset liquidation spirals, but would also fund such a relief program to mitigate fire-sale losses during a crisis.

The tax $T_i(\pi)$ is related to the systemicness of bank i, as defined in Greenwood et al. (2015): the amount a bank should be charged equals the component of its expected systemicness that is not borne by the bank itself. In other words, such a tax amount can be seen as the weighted average of banks' contributions to the aggregate vulnerability of the rest of the system over a number of stress tests with different initial market shocks. Cont and Schaanning (2017) describe stress tests in line with our model, as we assume that banks –rather than being leverage targeting like in Greenwood et al. (2015)– only sell assets if the leverage requirement is breached.

In the United States, the Financial Stability Oversight Council (FSOC) uses total consolidated assets, gross notional credit default swaps, derivative liabilities, total debt outstanding, leverage ratio, and short-term debt ratio as factors for designating systemically important financial institutions (SIFIs). An institution is designed as SIFI if these factors exceed certain thresholds.[7] Our study highlights another dimension to consider in the designation of SIFI institutions, in addition to too-big-to-fail measures of default costs. A highly central node in the network of asset holdings should be taxed more because it would cause higher disruption in the provision of services to the real economy during fire-sale events (e.g. interruption of project financing, and termination of productive investments due to suspension of loans).

6 Operational Challenges and Model Extensions

We discuss limitations of the current model and outline potential extensions as well as related operational challenges. Because our focus is on fire-sale spillovers, we ignore the possibility of a bank's default. If bank i's portfolio return falls below $-\frac{1}{1+\lambda_i}$, its equity becomes negative. A slight extension of the model would cap the amount a bank can liquidate to the total amount of assets held by the bank. Because the cost function is well-defined for all values of asset shocks Z, we assume the cost function to be uncapped.

In our model, the leverage ratio is updated only after the initial market shock, but not marked to market following price changes due to asset liquidation. While we assume that the initial shock on asset prices is permanent, the knocked-down effect on prices is only temporary. Marking to market would make leverage procyclical, because the initial asset liquidation –and not further fundamental market changes– would cause new rounds of deleveraging. We also remark that commercial banks in the US and universal banks in Europe do not mark the value of their assets to market.[8]

[7]The BIS has developed a methodology to identify systemically important financial institutions (SIFIs) based on asset size, interconnectedness, and the availability of substitutes for the services they provide.

[8]Regulators generally try to lean against mark-to-market valuation. One mechanism to enforce this is by requiring buffers in good times that can be run down in bad times. In other words, officials encourage banks to have

In our study, we assume that there is only one round of deleveraging. Because asset liquidation is costly, the revenue loss due to fire sales would result in the bank violating again the regulatory threshold, and thus trigger a new round of deleveraging. Hence, there would be infinite rounds of deleveraging. Realistically, banks are likely to target a leverage ratio that is strictly smaller than λ_M, and the resulting safety buffer protects the banks from subsequent rounds of deleveraging. To preserve tractability and highlight the main economic forces, we assume that banks target λ_M and perform only one round of asset liquidation. A similar assumption has also been made by Greenwood et al. (2015), who consider only the first round of deleveraging in their leverage targeting model.

Our model assumes that banks have full knowledge on the portfolio composition of other banks. This assumption is standard in existing literature on leverage targeting banks, in which the computation of systemic risk measures relies on publicly available data. In the U.S., financial institutions file form FR Y-9C with the Federal Reserve every quarter. These forms provide consolidated information on each bank's exposures and are available through the Board's Freedom of Information Office. Duarte and Eisenbach (2018) build their empirical model on FR Y-9C balance sheet data. The studies on the vulnerability of the European banking system by Greenwood et al. (2015), and Cont and Schaanning (2017) rely on publicly available data released by the European Banking Authority.

Banks are complex organizations consisting of different divisions, each with different incentives. Our framework is designed to capture the behavior of the asset management arm of a bank. Because such an arm consists of sophisticated traders who compete with asset manager arms of other banks, it is likely to have more granular information on the portfolio composition of its competitors.

7 Conclusions

Existing literature on fire sales has analyzed the mechanism through which hard balance sheet constraints and portfolio commonality exacerbate fire-sale externalities in the presence of distressed financial institutions. Our paper fills an important gap in the literature because it views banks as strategic as opposed to mechanical: Banks adjust their balance sheets to be more resilient to fire-sale spillovers. As such, our model does not simply provide a tool to study the propagation of financial contagion through the network of asset holdings due to overlapping portfolios. Rather, it sheds light on how fire-sale risk affects banks' ex-ante asset holding decisions. Furthermore, our model can be used to assess the welfare implications of government intervention as banks adapt to the new regulatory environment.

In the current paper, we focused on large leveraged institutions –typically banks– because of their significant contribution to systemic risk. Depending on their structure, financial institutions are subject to constraints that may force them to raise liquidity under adverse market conditions. For instance, hedge funds may need to liquidate assets if they face an approaching margin call; a

lower leverage in good times than in bad times. This is the purpose of the Basel III conservation buffer and the countercyclical buffer.

mutual fund may need to engage in massive sell-offs to pay its redeeming investors. Despite our focus on leverage constraints, our conclusions are far-reaching and may also be adapted to financial systems subject to other types of financial constraints.

A natural extension of the model includes assets with heterogeneous returns. Realistically, investments in illiquid assets are compensated with higher risk premia. In such an extension, banks are profit seeking and risk-averse. This would add another layer of complexity to the model, and may result in a loss of analytical tractability. Because such a model extension mimics more closely banks' balance sheet decision making, solving it –even if only numerically– would be of interest to regulators because it would allow them to better assess the consequences of supervisory intervention.

Furthermore, the model may be extended to study the dynamics of systemic diversification. Such a model would allow to analyze the relation between the dynamics of return shocks and the commonality-diversification trade-off. Decoupling portfolio holdings and liquidation strategy would allow to study the evolution of each bank's optimal portfolio over time and the resulting dynamics of fire-sale externalities.

A Technical Assumptions

First, we introduce *distance to liquidation* as a convenient reparameterization of the leverage ratio.

Definition A.1. *The distance to liquidation of bank i is $\ell_i := \frac{\lambda_M - \lambda_i}{(1+\lambda_i)\lambda_M}$, for $i = 1, \cdots, N$.*

Clearly, if banks satisfy the leverage requirement, there is a one-to-one relationship between leverage ratio and distance to liquidation. The latter quantity can be viewed as a rescaling of the leverage ratio: distance to liquidation is on the same scale as the portfolio return. A highly leveraged bank that breaches the leverage threshold even for a low decrease in its asset value has a low distance to liquidation. Vice versa, a bank with a low leverage ratio has a large distance to liquidation. Bank i is forced to liquidate assets to comply with the leverage requirement exactly when its portfolio return R_i at date 2 is below $-\ell_i$.

Assumption A.2. *For each $k = 1, \ldots, K$, Z_k has continuous probability density function, increasing on $[-\infty, 0]$, and the random vector Z is spherically symmetric.*

Assumption A.2 implies that all assets have the same distribution of returns. Such an assumption allows isolating the effect of fire sales from that of mean-variance optimization of returns. Furthermore, spherical symmetry guarantees that full diversification yields the lowest likelihood of liquidation. Because each bank's cost function only depends on the truncated distribution of its portfolio return, Assumption A.2 could be relaxed to also include probability density functions with asymmetric tails. This assumption is satisfied if Z is a centered Gaussian random vector, and the examples provided in the paper will be based on Gaussian returns.

Assumption A.3. *The potential function $P(\pi)$ is strictly convex on X^N.*

Assuming that the potential function is strictly convex ensures uniqueness of the equilibrium asset allocations. For $N = 2$, this assumption is implied by assumptions on the primitives of the model (see Theorem 3.2).

Assumption A.4. *For each bank i, $\ell_i \leq \bar{\ell}$ for a sufficiently small $\bar{\ell}$.*

Under Basel III, the required leverage constraint is $\lambda_M = 33$. This means that a leverage ratio of 20 implies a distance to liquidation equal to 1.9%. Even a leverage ratio of 10 implies a distance to liquidation of just 6.3%.

Assumption A.5. *The social planner's objective function $TC(\pi)$ is strictly convex on X^N.*

The assumption guarantees that the social planner admits a unique local minimum.

B Proofs

Lemma B.1. *If bank i's portfolio return is R_i, the amount x_i that bank i is required to raise is*

$$\lambda_M w_i (R_i + \ell_i)^-.$$

Proof. Bank i liquidates if $\frac{d_i}{w_i(1+R_i)-d_i} \geq \lambda_M$. After substituting $d_i = \frac{\lambda_i}{1+\lambda_i} w_i$, it can be seen that the inequality is equivalent to $R_i + \ell_i \leq 0$. Solving for x in the equation $\frac{d_i - x}{w_i(1+R_i)-d_i} = \lambda_M$, yields the quantity $x = \lambda_M w_i (R_i + \ell_i)^-$ that bank i is required to trade to comply with regulations. □

Proof of Proposition 3.1

Recall that the game is a potential game with potential function $P : X^N \to \mathbb{R}$ if $\forall i \in \{1, \cdots, N\}$, $\forall \pi_{-i} \in X^{N-1}$, $\forall \pi_i', \pi_i'' \in X$,

$$P(\pi_i', \pi_{-i}) - P(\pi_i'', \pi_{-i}) = \mathrm{EC}_i(\pi_i', \pi_{-i}) - \mathrm{EC}_i(\pi_i'', \pi_{-i}).$$

It can be immediately verified that $P(\pi)$ satisfies this condition.

If Z is a continuous random variable, then $P(\pi)$ is a continuous function over the compact set X^N. Hence, there exists $\pi^* \in X^N$ that minimizes $P(\pi)$. It can be verified that π^* is a Nash equilibrium. □

Proof of Theorem 3.2

Recall the definition of distance to liquidation ℓ_i in Definition A.1. First, we assume $\ell_1 = \ell_2$, $w_1 = w_2$ and $\gamma_1 = \gamma_2$, and prove that the potential function $P(\pi)$ is strongly convex. Notice that $\ell_1 = \ell_2$ is equivalent to $\lambda_1 = \lambda_2$.

With a slight abuse of notation, we denote $\pi_{i,1}$ simply by π_i, and hence $\pi_{i,2} = 1 - \pi_i$. We will show that the Hessian matrix H of $\frac{1}{\lambda_M^2 w^2 \gamma} P(\pi)$ is positive definite. For the first part of the proof we will consider any $N > 1$.

Define $A_i := \{\pi_i Z_1 + (1-\pi_i)Z_2 + \ell \le 0\}$ and $A_{i,j} := \{\pi_i Z_1 + (1-\pi_i)Z_2 + \ell \le 0, \pi_j Z_1 + (1-\pi_j)Z_2 + \ell \le 0\}$ for any $1 \le i, j \le N$. A simple calculation shows that $H_{i,i} = \frac{1}{\lambda_M^2 w^2 \gamma}\frac{\partial^2}{\partial \pi_i^2} P(\pi)$ is

$$E\Bigg[2(Z_1-Z_2)^2(\pi_i^2+(1-\pi_i)^2)1_{A_i} + 8(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)(Z_1-Z_2)(2\pi_i-1)1_{A_i} \\ + 4(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)^2 1_{A_i} + \sum_{j\neq i}\Big(2(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell_j)(Z_1-Z_2)(2\pi_j-1)1_{A_{i,j}} \\ + (\pi_i\pi_j + (1-\pi_i)(1-\pi_j))\frac{\partial}{\partial \pi_i}\left[(Z_1-Z_2)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell_j)1_{A_{i,j}}\right]\Big)\Bigg].$$

The off-diagonal element $H_{i,j} = \frac{1}{\lambda_M^2 w^2 \gamma}\frac{\partial^2}{\partial \pi_i \partial \pi_j} P(\pi)$ of the Hessian matrix H is

$$E\Bigg[(Z_1-Z_2)^2(\pi_i\pi_j + (1-\pi_i)(1-\pi_j))1_{A_{i,j}} + (\pi_j Z_1 + (1-\pi_j)Z_2 + \ell_j)(Z_1-Z_2)(2\pi_i-1)1_{A_{i,j}} \\ + (\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)(Z_1-Z_2)(2\pi_j-1)1_{A_{i,j}} \\ + 2(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell_j)1_{A_{i,j}}\Bigg].$$

Next, we construct a positive semidefinite matrix M_1 with the same off-diagonal elements as H. Define the random vectors $v^{(1)} := (\pi_i(Z_1-Z_2)1_{A_i})_{1\le i\le N}$, $v^{(2)} := ((1-\pi_i)(Z_1-Z_2)1_{A_i})_{1\le i\le N}$, $v^{(3)} := ((\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)1_{A_i})_{1\le i\le N}$, $v^{(4)} := ((Z_1-Z_2)(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell_i)1_{A_i})_{1\le i\le N}$ and $v^{(5)} := ((2\pi_i-1)1_{A_i})_{1\le i\le N}$. The random matrix $M_2 := v^{(1)}{v^{(1)}}^T + v^{(2)}{v^{(2)}}^T + 2v^{(3)}{v^{(3)}}^T + v^{(4)}{v^{(5)}}^T + v^{(5)}{v^{(4)}}^T$ is such that $E[M_2]$ has the same off-diagonal elements of H. Recall that for any couple of linearly independent vectors x, y, the matrix $xy^T + yx^T$ has exactly one negative eigenvalue, while the matrix xx^T has only one non-zero eigenvalue, which is positive. Now, we want to find a positive definite diagonal matrix D_1 such that $D_1 + M_2$ is almost surely positive semidefinite and the elements of D_1 are as small as possible. An immediate application of Woodbury matrix identity and the matrix determinant lemma shows that for any symmetric invertible matrix A, we have $\det(A + xy^T + yx^T) = ((1 + x^T A^{-1} y)^2 - (x^T A^{-1} x)(y^T A^{-1} y))\det(A)$. Therefore, $D_1 + v^{(4)}{v^{(5)}}^T + v^{(5)}{v^{(4)}}^T$ is positive semidefinite if and only if $a := (1 + {v^{(4)}}^T D_1^{-1} v^{(5)})^2 - ({v^{(4)}}^T D_1^{-1} v^{(4)})({v^{(5)}}^T D_1^{-1} v^{(5)}) \ge 0$. Define $b := \min_{\pi \in X^N} a$. It can be verified that the matrix D_1 with entries $d_{i,i}^{(1)} = \frac{N}{4}(2\ell + Z_1 + Z_2)^2 1_{A_i}$ is such that $b = 0$. Therefore, $D_1 + M_2$ is a positive semidefinite matrix.

We now show that

$$E\left[\frac{\partial}{\partial \pi_i}\left[(Z_1-Z_2)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)1_{A_{i,j}}\right]\right] > 0.$$

Assume that $\pi_i + \varepsilon < \pi_j$, for some small $\varepsilon > 0$, and define the event $A_{i,j}^{\varepsilon} := \{(\pi_i + \varepsilon)Z_1 + (1 - \pi_i - \varepsilon)Z_2 + \ell \le 0, \pi_j Z_1 + (1-\pi_j)Z_2 + \ell \le 0\}$. We have that $A_{i,j} \subset A_{i,j}^{\varepsilon}$ and that $Z_1 < Z_2$ on the event $A_{i,j}^{\varepsilon} \setminus A_{i,j}$. Therefore, $(Z_1 - Z_2)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell) > 0$ on $A_{i,j}^{\varepsilon} \setminus A_{i,j}$. Hence, for $\pi_i < \pi_j$, we have shown that the derivative is positive. Similarly, if $\pi_i > \pi_j$, $A_{i,j}^{\varepsilon} \subset A_{i,j}$ and $(Z_1 - Z_2)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell) < 0$ on $A_{i,j} \setminus A_{i,j}^{\varepsilon}$. It follows that also in this case the derivative is positive.

Hence, the diagonal matrix D_2 with elements

$$d_{i,i}^{(2)} = E\Bigg[\sum_{j\neq i}(\pi_i\pi_j + (1-\pi_i)(1-\pi_j))\frac{\partial}{\partial \pi_i}\left[(Z_1 - Z_2)(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)1_{A_{i,j}}\right]\Bigg]$$

is positive definite.

Next, we show that the diagonal matrix $M_3 := H - E[M_2 + D_1] - D_2$ is positive semidefinite. It follows then that H is positive definite. The i-th element on the diagonal of M_3 is

$$\begin{aligned} E\Bigg[&(Z_1 - Z_2)^2(\pi_i^2 + (1-\pi_i)^2)1_{A_i} + 6(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1)1_{A_i} \\ &+ 2(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)^2 1_{A_i} - \frac{N(Z_1 + Z_2 + 2\ell)^2}{4}1_{A_i} \\ &+ \sum_{j\neq i} 2(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1)1_{A_{i,j}}\Bigg]. \end{aligned}$$

Assume now $N = 2$. First, we prove that $E\left[\left((Z_1 - Z_2)^2(\pi_i^2 + (1-\pi_i)^2) - 2(\frac{Z_1}{2} + \frac{Z_2}{2} + \ell)^2)\right)1_{A_i}\right] > 0$. Define $d_{ax+by}(z_1, z_2) := \frac{|az_1 + bz_2|}{(a^2+b^2)^{1/2}}$ the distance of the point (z_1, z_2) to the line $ax + by = 0$. The inequality can then be rephrased as $E\left[\left(2d_{y-x}^2(Z_1, Z_2)(\pi_i^2 + (1-\pi_i)^2) - d_{x/2+y/2+\ell}^2(Z_1, Z_2)\right)1_{A_i}\right] > 0$. Since $\pi_i^2 + (1-\pi_i)^2 \geq \frac{1}{2}$, it is enough to prove that $E\left[\left(d_{y-x}^2(Z_1, Z_2) - d_{x/2+y/2+\ell}^2(Z_1, Z_2)\right)1_{A_i}\right] > 0$. Define $A_i^{(1)} = A_i \cap \{\pi_i Z_1 - (1-\pi_i)Z_2 + (2\pi_i - 1)\ell < 0\}$ and $A_i^{(2)} = A_i \cap \{\pi_i Z_1 - (1-\pi_i)Z_2 + (2\pi_i - 1)\ell \geq 0\}$. Notice that the line $\pi_i x - (1-\pi_i)y + (2\pi_i - 1)\ell = 0$ is the reflection of the line $\pi_i x + (1-\pi_i)y + \ell = 0$ over $y = -\ell$. We will show that $E\left[\left(d_{y-x}^2(Z_1, Z_2) - d_{x/2+y/2+\ell}^2(Z_1, Z_2)\right)1_{A_i^{(1)}}\right] > 0$, the case for $A_i^{(2)}$ is analogous. In the set $A_i^{(1)}$, consider the points $(x_1, y_1) = (z_1, z_2)$ and $(x_2, y_2) = (z_1, -2\ell - z_2)$ with $z_2 > -\ell$, which are symmetric with respect to the line $y = -\ell$. They are such that $d_{y-x}(x_1, y_1) > d_{x/2+y/2+\ell}(x_1, y_1)$, $d_{y-x}(x_1, y_1) = d_{x/2+y/2+\ell}(x_2, y_2)$ and $d_{y-x}(x_2, y_2) = d_{x/2+y/2+\ell}(x_1, y_1)$. In particular,

$$(d_{y-x}^2(x_1, y_1) - d_{x/2+y/2+\ell}^2(x_1, y_1)) + (d_{y-x}^2(x_2, y_2) - d_{x/2+y/2+\ell}^2(x_2, y_2)) = 0.$$

Notice also that $\|(x_1, y_1)\| < \|(x_2, y_2)\|$. Since the distribution of Z is rotationally invariant and Z_1 has an increasing probability density function on $(-\infty, 0]$, we have $\varphi(x_1, y_1) > \varphi(x_2, y_2)$, where $\varphi(\cdot,\cdot)$ is the probability density function of (Z_1, Z_2). Therefore,

$$\begin{aligned} E\left[\left(d_{y-x}^2(Z_1, Z_2) - d_{x/2+y/2+\ell}^2(Z_1, Z_2)\right)1_{A_i^{(1)}}\right] &= \int_{A_i^{(1)}}(d_{y-x}^2(z_1, z_2) - d_{x/2+y/2+\ell}^2(z_1, z_2))\varphi(z_1, z_2)dz_1dz_2 \\ = \int_{A_i^{(1)} \cap \{z_2 > -\ell\}}&(d_{y-x}^2(z_1, z_2) - d_{x/2+y/2+\ell}^2(z_1, z_2))(\varphi(z_1, z_2) - \varphi(z_1, -2\ell - z_2))dz_1dz_2. \end{aligned}$$

Both terms in the product of the last integrand are positive. It follows that the expectation on $A_i^{(1)}$ is positive. The same arguments hold for the expectation on $A_i^{(2)}$, with $(x_1, y_1) = (z_1, z_2)$ and $(x_2, y_2) = (-2\ell - z_1, z_2)$ where $z_1 > -\ell$.

Next, we prove that

$$E\left[(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1)1_{A_i} + (\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1)1_{A_{i,j}}\right] > 0.$$

Assume that $\pi_i > 1/2$, $\pi_j < 1/2$ and $|\pi_i - 1/2| < |\pi_j - 1/2|$ (analogous arguments apply for all other combinations). Notice that $(\pi_i z_1 + (1-\pi_i)z_2 + \ell)(z_1 - z_2)(2\pi_i - 1) > 0$ for any $(z_1, z_2) \in A_i \setminus A_{i,j}$. Hence, it is enough to show $E[((\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1) + (\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1))1_{A_{i,j}}] \geq 0$. Define $A_{i,j}^{(1)} = A_{i,j} \cap \{(1-\pi_j)Z_1 + \pi_j Z_2 + \ell < 0\}$ and $A_{i,j}^{(2)} = A_{i,j} \cap \{(1-\pi_j)Z_1 + \pi_j Z_2 + \ell > 0\}$. On $A_{i,j}^{(2)}$ we have $d_{\pi_i x + (1-\pi_i)y + \ell} < d_{\pi_j x + (1-\pi_j)y + \ell}$. Therefore, for any $(z_1, z_2) \in A_{i,j}^{(2)}$, since $(1+x^2)^{1/2}x$ is increasing on $[0,1]$, we get

$$\begin{aligned}
&(\pi_i z_1 + (1-\pi_i)z_2 + \ell)(2\pi_i - 1) + (\pi_j z_1 + (1-\pi_j)z_2 + \ell)(2\pi_j - 1) = \\
&\quad -\frac{1}{\sqrt{2}} d_{\pi_i x + (1-\pi_i)y + \ell}(z_1, z_2)(1 + (2\pi_i - 1)^2)^{1/2}(2\pi_i - 1) \\
&\quad + \frac{1}{\sqrt{2}} d_{\pi_j x + (1-\pi_j)y + \ell}(z_1, z_2)(1 + (2\pi_j - 1)^2)^{1/2}|2\pi_j - 1| > \\
&\quad \frac{1}{\sqrt{2}}(-d_{\pi_i x + (1-\pi_i)y + \ell}(z_1, z_2) + d_{\pi_j x + (1-\pi_j)y + \ell}(z_1, z_2))(1 + (2\pi_i - 1)^2)^{1/2}(2\pi_i - 1) > 0.
\end{aligned}$$

Since $z_1 - z_2 > 0$ on $A_{i,j}^{(2)}$, we get $E[((\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1) + (\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1))1_{A_{i,j}^{(2)}}] > 0$. Next, we show that both $E[(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1)1_{A_{i,j}^{(1)}}]$ and $E[(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1))1_{A_{i,j}^{(1)}}]$ are positive. Consider the point $(z_1, z_2) \in A_{i,j}^{(1)}$ with $z_2 > z_1$, then also $(z_2, z_1) \in A_{i,j}^{(1)}$. Since $d_{\pi_i x + (1-\pi_i)y + \ell}(z_1, z_2) > d_{\pi_i x + (1-\pi_i)y + \ell}(z_2, z_1)$, we get $(\pi_i z_1 + (1-\pi_i)z_2 + \ell)(z_1 - z_2)(2\pi_i - 1) + (\pi_i z_2 + (1-\pi_i)z_1 + \ell)(z_2 - z_1)(2\pi_i - 1) > 0$. It follows that $E[(\pi_i Z_1 + (1-\pi_i)Z_2 + \ell)(Z_1 - Z_2)(2\pi_i - 1)1_{A_{i,j}^{(1)}}] > 0$. Similarly, it can be shown that $E[(\pi_j Z_1 + (1-\pi_j)Z_2 + \ell)(Z_1 - Z_2)(2\pi_j - 1)1_{A_{i,j}^{(1)}}] > 0$.

The remaining terms in the diagonal elements of M_3 are positive. Hence, H is positive definite for all $(\pi_1, \pi_2) \in [0,1]^2$, and therefore $P(\pi)$ is strongly convex.

Because of the smoothness of the potential function with respect to the parameters, there exist $\ell_* < \ell < \ell^*$ (equivalently, $\lambda_* < \lambda < \lambda^*$), $w_* < w < w^*$, $\gamma_* < \gamma < \gamma^*$ such that if $\ell_i \in (\ell_*, \ell^*)$ (equivalently, if $\lambda_i \in (\lambda_*, \lambda^*)$), $w_i \in (w_*, w^*)$ for each i and $\gamma_k \in (\gamma_*.\gamma^*)$ for each k, then $P(\pi)$ is strictly convex on X^N. The uniqueness of the Nash equilibrium then follows from Theorem 2.3 in Lã et al. (2016). □

Proof of Proposition 4.1

Recall the definition of distance to liquidation ℓ in Definition A.1. By Lemma B.1, the bank's expected liquidation costs are given by $\lambda_M^2 w^2 \gamma \|\pi\|_2^2 E[(\pi^T Z + \ell)^2 1_{\{\pi^T Z + \ell \leq 0\}}]$. Since Z is spherically symmetric, $\frac{1}{\|\pi\|_2}\pi^T Z$ has the same distribution as Z_1. Hence, $\|\pi\|_2^2 E[(\pi^T Z + \ell)^2 1_{\{\pi^T Z+\ell\leq 0\}}] = \|\pi\|_2^2 E[(\|\pi\|_2 Z_1 + \ell)^2 1_{\{\|\pi\|_2 Z_1+\ell\leq 0\}}]$. It follows that the expected liquidation costs are minimized when the bank minimizes $\|\pi\|_2$. The minimum of $\|\pi\|_2$ is attained at $\pi_k = \frac{1}{K}$ for each k. □

Proof of Proposition 4.2

Recall the definition of distance to liquidation ℓ in Definition A.1. Define $f(x,\ell) := E[(xZ_1 + (1-x)Z_2 + \ell)^2 1_{\{xZ_1+(1-x)Z_2+\ell\leq 0\}}]$ and $g(x) := x^2\gamma_1 + (1-x)^2\gamma_2$. The minimizer of $f(\cdot,\ell)$ is $\frac{1}{2}$ and the minimizer of $g(\cdot)$ is $x_g := \frac{\gamma_2}{\gamma_1+\gamma_2} > \frac{1}{2}$. We write $f_x(x,\ell)$ for $\frac{\partial}{\partial x}f(x,\ell)$. Since $\frac{d}{d\pi_{1,1}}S_1 = f_x(\pi_{1,1})g(\pi_{1,1}) + f(\pi_{1,1})g_x(\pi_{1,1})$, $f \geq 0$, $g > 0$, $f_x(x) < 0$ on $[0,\frac{1}{2})$ and $f_x(x) > 0$ on $(\frac{1}{2},1]$, and $g_x(x) < 0$ on $[0,x_g)$ and $g_x(x) > 0$ on $(x_g,1]$, we get that $\frac{d}{d\pi_{1,1}}S_1 < 0$ on $[0,\frac{1}{2}]$ and $\frac{d}{d\pi_{1,1}}S_1 > 0$ on $[x_g,1]$. Hence, $\pi_{1,1}^S \in (\frac{1}{2},\frac{\gamma_2}{\gamma_1+\gamma_2})$.

Next, we show that $\frac{f_x(x,\ell)}{f(x,\ell)}$ is an increasing function of ℓ for $x > \frac{1}{2}$. Since Z is rotationally invariant, $f(x) = E[(n(x)Z_1 + \ell)^2 1_{\{n(x)Z_1+\ell\leq 0\}}]$, where $n(x) = (x^2 + (1-x)^2)^{1/2}$. An explicit calculation shows that $f_{x\ell}(\pi_{1,1})f(\pi_{1,1}) - f_x(\pi_{1,1})f_\ell(\pi_{1,1})$ is equal to

$$-\frac{2(2\pi_{1,1}-1)}{n(\pi_{1,1})}(2E[Z_1^2 1_A]P(A)\ell n(\pi_{1,1}) + \ell^2 E[Z_1 1_A]P(A) + E[Z_1 1_A]E[Z_1^2 1_A]n(\pi_{1,1})^2),$$

where $A = \{n(\pi_{1,1})Z_1 + \ell \leq 0\}$. Because the distribution of Z_1 is increasing on $(-\infty, 0]$, we get $\frac{\ell}{n(\pi_{1,1})}P(A) + E[Z_1 1_A] < 0$ and $E[Z_1^2 1_A] + \frac{\ell}{n(\pi_{1,1})}E[Z_1 1_A] < 0$. It follows that $f_{x\ell}(\pi_{1,1})f(\pi_{1,1}) - f_x(\pi_{1,1})f_\ell(\pi_{1,1}) > 0$ for $\pi_{1,1} > \frac{1}{2}$. Therefore, if $\ell_2 > \ell_1$ we have $\frac{f_x(\pi_{1,1}^S(\ell_1),\ell_2)}{f(\pi_{1,1}^S(\ell_1),\ell_2)} > \frac{f_x(\pi_{1,1}^S(\ell_1),\ell_1)}{f(\pi_{1,1}^S(\ell_1),\ell_1)} = \frac{g_x(\pi_{1,1}^S(\ell_1))}{g(\pi_{1,1}^S(\ell_1))}$. In other terms, $\frac{d}{d\pi_{1,1}}S_1(\pi^S(\ell_1),\ell_2) > 0$. Since $\frac{d}{d\pi_{1,1}}S_1(\pi_{1,1},\ell)$ is increasing in $\pi_{1,1}$, we get that $\pi_{1,1}^S(\ell_2) < \pi_{1,1}^S(\ell_1)$. Because ℓ is a decreasing function of λ, we obtain the thesis. □

Proof of Proposition 4.3

Rewrite the allocation vector $\pi_i \in X$ of bank i as $\left(\pi_{i,1},\cdots,\pi_{i,K-1}, 1-\sum_{k=1}^{K-1}\pi_{i,k}\right)$. Using Lemma B.1, we get that the derivative $\frac{1}{\lambda_M^2}\frac{\partial}{\partial \pi_{i,h}}P(\pi)$ is

$$\begin{aligned}
E\Bigg[&2w_i^2(\pi_i^T Z + \ell_i)\pi_i^T \mathrm{Diag}(\gamma)\pi_i(Z_h - Z_K)1_{A_i} + 2w_i^2(\pi_i^T Z + \ell_i)^2(\pi_{i,h}\gamma_h - (1-\sum_{k=1}^{K-1}\pi_{i,k})\gamma_K)1_{A_i} + \\
&\sum_{j\neq i} w_i w_j(\pi_j^T Z + \ell_j)\pi_i^T \mathrm{Diag}(\gamma)\pi_j(Z_h - Z_K)1_{A_{i,j}} + \\
&\sum_{j\neq i} w_i w_j(\pi_i^T Z + \ell_i)(\pi_j^T Z + \ell_j)(\pi_{j,h}\gamma_h - (1-\sum_{k=1}^{K-1}\pi_{j,k})\gamma_K)1_{A_{i,j}}\Bigg].
\end{aligned}$$

To prove (1), assume that $\gamma_k = \gamma$ and $\pi_{i,k} = \frac{1}{K}$ for all i and k. Because Z is spherically symmetric, $E[(\sum_{k=1}^K Z_k + K\ell_i)Z_h 1_{\{\sum_{k=1}^K Z_k+K\ell_i\leq 0\}}] = E[(\sum_{k=1}^K Z_k + K\ell_i)Z_K 1_{\{\sum_{k=1}^K Z_k+K\ell_i\leq 0\}}]$ for each h. Hence, the first term in the derivative is 0. The second term is 0, because one of its coefficients is 0. The same arguments yield that the third and fourth term are also 0. Since the potential function

$P(\pi)$ is convex, $\pi_{i,k} = \frac{1}{K}$ for all i and k is the unique Nash equilibrium.

To prove (2), first notice that π^* is a critical point for $S_i(\pi_i)$. It follows that π^* solves the equations $E[(\pi^{*T}Z+\ell)\pi^{*T}\text{Diag}(\gamma)\pi_i^*(Z_h-Z_K)1_{A_i}+(\pi^{*T}Z+\ell)^2(\pi_h^*\gamma_h-(1-\sum_{k=1}^{K-1}\pi_k^*)\gamma_K)1_{A_i}] = 0$ for every h. If $\ell_i = \ell_j = \ell$ and $\pi_i = \pi_j = \pi^*$, then $A_{i,j} = A_i$. Hence, the sum of the first two terms in the derivative of $P(\pi)$ and the sum of the last two terms are both 0. From the convexity of $P(\pi)$ it follows that $\pi_i = \pi^*$ for each i is the unique Nash equilibrium. □

Lemma B.2. *Assume $K = 2$. Under Assumptions A.2 and A.4, $\frac{\partial^2 P}{\partial \pi_{i,1}\pi_{j,1}}(\pi) > 0$ for $1 \le i \ne j \le N$ and $\pi \in X^N$.*

Proof. A simple calculation shows that $\frac{1}{\lambda_M^2}\frac{\partial^2 P}{\partial \pi_{i,1}\pi_{j,1}}(\pi) = E[(\gamma_1(\ell_i + 2\pi_{i,1}Z_1 + (1-2\pi_{i,1})Z_2)(\ell_j + 2\pi_{j,1}Z_1+(1-2\pi_{j,1})Z_2)+\gamma_2(\ell_i+(2\pi_{i,1}-1)Z_1+2(1-\pi_{i,1})Z_2)(\ell_j+(2\pi_{j,1}-1)Z_1+2(1-\pi_{j,1})Z_2))1_{A_i\cap A_j}]$, where $A_i := \{\pi_{1,1}Z_1 + (1-\pi_{i,1})Z_2 + \ell_i \le 0\}$ and $A_j := \{\pi_{j,1}Z_1 + (1-\pi_{j,1})Z_2 + \ell_j \le 0\}$. Assume $\ell_i = \ell_j = 0$. From the spherical symmetry of the distribution of Z, it follows that the distribution is uniform along every circle. Next, we consider the change of variable $(Z_1, Z_2) = (\rho\cos(t), \rho\sin(t))$. In terms of the new variables (ρ, t), if $\pi_{i,1} > \pi_{j,1}$, the integration region $A_i \cap A_j$ translates into the range $(0,+\infty)\times[t_l, t_r] := (0,+\infty)\times\left[\arctan\left(-\frac{\pi_{j,1}}{1-\pi_{j,1}}\right)+\pi, \arctan\left(-\frac{\pi_{i,1}}{1-\pi_{i,1}}\right)+2\pi\right]$. For a fixed ρ, the integral over t reduces to

$$\begin{aligned}&\gamma_1(2(t_r-t_l)+2(-(\pi_{i,1}+\pi_{j,1})+4\pi_{i,1}\pi_{j,1})(2t_r+\cos(2t_r)-2t_l-\cos(2t_l))\\&+(2(\pi_{i,1}+\pi_{j,1})-1)(\sin(2t_r)-\sin(2t_l)))+\gamma_2(\cdots),\end{aligned}$$

up to a positive coefficient. The expression that multiplies γ_1 is strictly positive. The same calculations and arguments hold for the term that multiplies γ_2. Hence, the integral over ρ is also strictly positive. By continuity, the expectation is strictly positive for $\ell_1, \ell_2 \le \bar{\ell}$, for a sufficiently small $\bar{\ell}$. □

Proof of Theorem 4.4

With a slight abuse of notation, we denote $\pi_{i,1}$ simply by π_i, and hence $\pi_{i,2} = 1-\pi_i$. The Nash equilibrium (π_1^*, π_2^*) solves the system of equations

$$\begin{aligned}\frac{1}{\lambda_M^2}\frac{\partial}{\partial \pi_i}P(\pi_i,\pi_j) = E\Big[&2w_i^2(\pi_iZ_1+(1-\pi_i)Z_2+\ell_i)(\gamma_1\pi_i^2+\gamma_2(1-\pi_i)^2)(Z_1-Z_2)1_{A_i} \\ &+2w_i^2(\pi_iZ_1+(1-\pi_i)Z_2+\ell_i)^2((\gamma_1+\gamma_2)\pi_i-\gamma_2)1_{A_i}\\ &+w_iw_j(\pi_jZ_1+(1-\pi_j)Z_2+\ell_j)(\gamma_1\pi_i\pi_j+\gamma_2(1-\pi_i)(1-\pi_j))(Z_1-Z_2)1_{A_{i,j}}\\ &+w_iw_j(\pi_iZ_1+(1-\pi_i)Z_2+\ell_i)(\pi_jZ_1+(1-\pi_j)Z_2+\ell_j)((\gamma_1+\gamma_2)\pi_j-\gamma_2)1_{A_{i,j}}\Big]=0\\ &\text{for } i=1, j=2 \text{ and } i=2, j=1,\end{aligned} \tag{B.1}$$

where $A_i := \{\pi_1Z_1+(1-\pi_i)Z_2+\ell_i \le 0\}$, $A_j := \{\pi_jZ_1+(1-\pi_j)Z_2+\ell_j \le 0\}$, and $A_{i,j} := A_i \cap A_j$. If $\ell_2 = \ell_1$, then there exists $\pi^{\ell_1} \in (\frac{1}{2}, \frac{\gamma_1+\gamma_2}{\gamma_2})$ such that $\pi_1 = \pi_2 = \pi^{\ell_1}$ is the Nash equilibrium. In particular, since $\frac{\partial}{\partial \pi_i} S_i(\pi_i) = 0$ for $\pi_i = \pi^{\ell_1}$, both the sum of the first two terms, i.e., $\frac{\partial}{\partial \pi_i} S_i(\pi_i)$, and the sum of the last two terms, i.e., $\frac{\partial}{\partial \pi_i} M_{i,j}(\pi_i, \pi_j)$, are zero. Consider equation (B.1) where $i = 1, j = 2$. For $\ell_1 < \ell_2$ and $\pi_1 = \pi_2 = \pi^{\ell_1}$, we will show that the sum of the last two terms is negative, i.e., $\frac{\partial}{\partial \pi_i} M_{i,j}(\pi_i, \pi_j) < 0$. This is equivalent to proving that

$$\frac{E\left[(\pi^{\ell_1} Z_1 + (1-\pi^{\ell_1})Z_2 + \ell_2)(Z_1 - Z_2)1_{A_{1,2}}\right]}{E\left[(\pi^{\ell_1} Z_1 + (1-\pi^{\ell_1})Z_2 + \ell_1)(\pi^{\ell_1} Z_1 + (1-\pi^{\ell_1})Z_2 + \ell_2)1_{A_{1,2}}\right]} <$$

$$-\frac{((\gamma_1+\gamma_2)\pi^{\ell_1} - \gamma_2)}{(\gamma_1 {\pi^{\ell_1}}^2 + \gamma_2(1-\pi^{\ell_1})^2)} = \frac{E\left[(\pi^{\ell_1} Z_1 + (1-\pi^{\ell_1})Z_2 + \ell_1)(Z_1 - Z_2)1_{A_1}\right]}{E\left[(\pi^{\ell_1} Z_1 + (1-\pi^{\ell_1})Z_2 + \ell_1)^2 1_{A_1}\right]},$$

where the last equality follows from $\frac{\partial}{\partial \pi_1} S_1(\pi_1) = 0$ for $\pi_1 = \pi^{\ell_1}$. Define $h(\ell_2) := E[(\pi^{\ell_1} Z_1 + (1-\pi^{\ell_1})Z_2 + \ell_2)(Z_1 - Z_2)1_{A_{1,2}}]$ and $k(\ell_2) := E[(\pi^{\ell_1} Z_1 + (1-\pi^{\ell_1})Z_2 + \ell_1)(\pi^{\ell_1} Z_1 + (1-\pi^{\ell_1})Z_2 + \ell_2)1_{A_{1,2}}]$. The inequality we want to prove can thus be rewritten as $\frac{h(\ell_2)}{k(\ell_2)} < \frac{h(\ell_1)}{k(\ell_1)}$. Notice that $h(\ell_2) = \frac{1}{2}\frac{\partial}{\partial \pi} E[(\pi Z_1 + (1-\pi)Z_2 + \ell_2)^2 1_{A_{1,2}}]|_{\pi=\pi^{\ell_1}} = \frac{1}{2}\frac{\partial}{\partial \pi} E[(n(\pi)Z_1 + \ell_2)^2 1_{A_{1,2}}]|_{\pi=\pi^{\ell_1}} = E[(n(\pi^{\ell_1})Z_1 + \ell_2)\frac{2\pi^{\ell_1}-1}{n(\pi^{\ell_1})} Z_1 1_{A_{1,2}}]$, where $n(\pi) = (\pi^2 + (1-\pi)^2)^{1/2}$. Also, $k(\ell_2) = E[(n(\pi^{\ell_1})Z_1 + \ell_1)(n(\pi^{\ell_1})Z_1 + \ell_2)1_{A_{1,2}}]$. Explicit computations show that $h'(\ell_2)k(\ell_2) - h(\ell_2)k'(\ell_2) = (2\pi^{\ell_1} - 1)\ell_1(E[Z_1 1_{A_{1,2}}]^2 - E[Z_1^2 1_{A_{1,2}}]P(A_{1,2}))$. It follows from the Cauchy-Schwartz inequality and $\pi^{\ell_1} > \frac{1}{2}$ that this expression is negative. Hence, the fraction $\frac{h(\ell)}{k(\ell)}$ is decreasing in ℓ, which proves the inequality. In particular, it follows that $\frac{\partial}{\partial \pi_1} P(\pi_1, \pi_2) < 0$ for $(\pi_1, \pi_2) = (\pi^{\ell_1}, \pi^{\ell_1})$. From the convexity of the potential function $P(\pi_1, \pi_2)$, we get that $\frac{\partial}{\partial \pi_1} P(\pi_1, \pi_2) = 0$ for $(\pi_1, \pi_2) = (\pi_1^{(0)}, \pi^{\ell_1})$, where $\pi_1^{(0)} > \pi^{\ell_1}$. Similarly, $\frac{\partial}{\partial \pi_2} P(\pi_1, \pi_2) = 0$ for $(\pi_1, \pi_2) = (\pi^{\ell_2}, \pi_2^{(0)})$, where $\pi_2^{(0)} < \pi^{\ell_2}$. In particular, $\pi_2^{(0)} < \pi^{\ell_2} < \pi^{\ell_1} < \pi_1^{(0)}$, where the second inequality follows from Proposition 4.2.

Next, we show that the optimal weight π_1 of bank 1 is a decreasing function of π_2. For a fixed π_2, let $\bar{\pi}_1(\pi_2)$ be the optimal response by bank 1, i.e., $\frac{\partial}{\partial \pi_1} P(\bar{\pi}_1(\pi_2), \pi_2) = 0$. Differentiating both the left and right hand side with respect to π_2 yields $\frac{\partial^2}{\partial \pi_1 \partial \pi_2} P(\bar{\pi}_1(\pi_2), \pi_2) + \frac{\partial^2}{\partial \pi_1^2} P(\bar{\pi}_1(\pi_2), \pi_2) \times \frac{\partial}{\partial \pi_2}\bar{\pi}_1 = 0$. From the convexity of P and Lemma B.2, it follows that $\bar{\pi}_1(\pi_2)$ is decreasing, and therefore $\bar{\pi}_1^{-1}(\pi_1)$ is decreasing. Analogously, $\bar{\pi}_2(\pi_1)$ is a decreasing function.

We have already proved that $\bar{\pi}_1(\pi^{\ell_1}) = \pi_1^{(0)}$ and $\bar{\pi}_2(\pi^{\ell_2}) = \pi_2^{(0)}$, where $\pi_2^{(0)} < \pi^{\ell_2} < \pi^{\ell_1} < \pi_1^{(0)}$. Notice that $\bar{\pi}_1^{-1}(\pi_1^{(0)}) = \pi^{\ell_1} > \pi_2^{(0)} = \bar{\pi}_2(\pi^{\ell_2}) > \bar{\pi}_2(\pi_1^{(0)})$. Define $\pi_{1,r} := \bar{\pi}_1(0)$. It follows that $0 = \bar{\pi}_1^{-1}(\pi_{1,r}) \leq \bar{\pi}_2(\pi_{1,r})$. By continuity, there exists $\pi_1^* \in (\pi_1^{(0)}, \pi_{1,r}]$ such that $\bar{\pi}_1^{-1}(\pi_1^*) = \bar{\pi}_2(\pi_1^*)$. The point $(\pi_1^*, \bar{\pi}_2(\pi_1^*))$ is by definition the Nash equilibrium. Because $\pi_1^* > \pi^{\ell_1}$ and $\bar{\pi}_2(\pi_1^*) < \bar{\pi}_2(\pi^{\ell_2}) = \pi_2^{(0)} < \pi^{\ell_2}$, we get the thesis. $\square$

Proof of Theorem 5.1

Notice that $TC(\pi) = 2P(\pi) - S(\pi)$. From the proof of Theorem 4.4 we know that $\frac{\partial}{\partial \pi_1}(2P - S)(\pi_1^{(0)}, \pi^{\ell_1}) < 0$. From the convexity of $TC(\cdot)$, it follows that there exists a unique $\pi_1^{(1)}$ such that

$\frac{\partial}{\partial \pi_1} TC(\pi_1^{(1)}, \pi^{\ell_1}) = 0$, where $\pi_1^{(1)} > \pi_1^{(0)}$. Similarly, $\frac{\partial}{\partial \pi_2} TC(\pi^{\ell_2}, \pi_2^{(1)}) = 0$ for $\pi_2^{(1)} < \pi_2^{(0)}$. Hence, $\pi_2^{(1)} < \pi_2^{(0)} < \pi^{\ell_2} < \pi^{\ell_1} < \pi_1^{(0)} < \pi_1^{(1)}$.

Given π_2, let $\bar{\pi}_1^{SP}(\pi_2)$ be the minimizer of $TC(\cdot, \pi_2)$, i.e., $\frac{\partial}{\partial \pi_1} TC(\bar{\pi}_1^{SP}(\pi_2), \pi_2) = 0$. As in the proof of Theorem 4.4, differentiating the left and right hand side with respect to π_2 yields $\frac{\partial^2}{\partial \pi_1 \partial \pi_2} TC(\bar{\pi}_1^{SP}(\pi_2), \pi_2) + \frac{\partial^2}{\partial \pi_1^2} TC(\bar{\pi}_1^{SP}(\pi_2), \pi_2) \times \frac{\partial}{\partial \pi_2} \bar{\pi}_1^{SP} = 0$. It follows that

$$\frac{\partial}{\partial \pi_2} \bar{\pi}_1^{SP} = -\frac{\frac{\partial^2}{\partial \pi_1 \partial \pi_2} TC(\bar{\pi}_1^{SP}(\pi_2), \pi_2)}{\frac{\partial^2}{\partial \pi_1^2} TC(\bar{\pi}_1^{SP}(\pi_2), \pi_2)} = -\frac{\frac{\partial^2}{\partial \pi_1 \partial \pi_2} P(\bar{\pi}_1^{SP}(\pi_2), \pi_2)}{\frac{\partial^2}{\partial \pi_1^2} (P - 1/2S)(\bar{\pi}_1^{SP}(\pi_2), \pi_2)} < \frac{\partial}{\partial \pi_2} \bar{\pi}_1 < 0.$$

Therefore, $0 > \frac{\partial}{\partial \pi_1} (\bar{\pi}_1^{SP})^{-1} > \frac{\partial}{\partial \pi_1} \bar{\pi}_1^{-1}$. Since $\bar{\pi}_2^{SP}(\pi^{\ell_2}) = \pi_2^{(1)} < \pi_2^{(0)} = \bar{\pi}_2(\pi^{\ell_2})$ and $\frac{\partial}{\partial \pi_2} \bar{\pi}_1^{SP} < \frac{\partial}{\partial \pi_2} \bar{\pi}_1$, we get $\bar{\pi}_2^{SP} < \bar{\pi}_2$ on $[\pi^{\ell_2}, 1]$. Analogously, $(\bar{\pi}_1^{SP})^{-1} > \bar{\pi}_1^{-1}$ on $[\pi^{\ell_1}, \pi_{1,r}^{SP}]$, where $\pi_{1,r}^{SP} := \bar{\pi}_1^{SP}(0)$. By continuity, there exists $\pi_1^{*,SP} \in (\pi_1^{(1)}, \pi_{1,r}^{SP}]$ such that $(\bar{\pi}_1^{SP})^{-1}(\pi_1^{*,SP}) = \bar{\pi}_2^{SP}(\pi_1^{*,SP})$ and $\pi_1^{*,SP} > \pi_1^*$. This completes the proof of the theorem. $\square$

Proof of Proposition 5.3

It is enough to observe that $EC_i(\pi) + T_i(\pi) - TC(\pi)$ does not depend on π_i. Hence, for any π_{-i} the allocation π_i that minimizes $EC_i(\pi_i, \pi_{-i}) + T_i(\pi_i, \pi_{-i})$ also minimizes $TC(\pi_i, \pi_{-i})$. $\square$

Proof of Proposition 5.2

By symmetry, $TC^*(d/2 - \varepsilon) = TC^*(d/2 + \varepsilon)$. It follows that $\frac{d}{2}$ is a critical point for $TC^*(\cdot)$.

Notice that if ℓ (resp. ℓ_1) is the distance to liquidation for the bank with asset value w and debt d (resp. $w_1 := \frac{w}{2}$ and d_1), then a bank with asset value $w_2 := \frac{w}{2}$ and debt $d_2 := d - d_1$ has distance to liquidation $\ell_2 := 2\ell - \ell_1$. Let $\pi_1^*(\ell_1)$ and $\pi_2^*(\ell_1)$ be the allocations in equilibrium for bank 1 and bank 2 in a system with two banks of equal size $\frac{w}{2}$ and distance of liquidation ℓ_1 and ℓ_2. Next, we prove that $\frac{\partial}{\partial \ell_1} \pi_1^*|_{\ell_1 = \ell} = -\frac{\partial}{\partial \ell_1} \pi_2^*|_{\ell_1 = \ell}$. Because $\pi_1^*(\ell_1)$ and $\pi_2^*(\ell_1)$ solve the system of equations $\frac{\partial}{\partial \pi_1} P(\pi_1, \pi_2) = 0$, $\frac{\partial}{\partial \pi_2} P(\pi_1, \pi_2) = 0$, differentiating with respect to ℓ_1 yields

$$\begin{aligned} &\frac{\partial^2}{\partial \pi_1 \partial \ell_1} P(\pi_1^*, \pi_2^*) + \frac{\partial^2}{\partial \pi_1^2} P(\pi_1^*, \pi_2^*) \frac{\partial}{\partial \ell_1} \pi_1^* + \frac{\partial^2}{\partial \pi_1 \partial \pi_2} P(\pi_1^*, \pi_2^*) \frac{\partial}{\partial \ell_1} \pi_2^* = 0, \qquad \text{(B.2)} \\ &\frac{\partial^2}{\partial \pi_2 \partial \ell_1} P(\pi_1^*, \pi_2^*) + \frac{\partial^2}{\partial \pi_1 \partial \pi_2} P(\pi_1^*, \pi_2^*) \frac{\partial}{\partial \ell_1} \pi_1^* + \frac{\partial^2}{\partial \pi_2^2} P(\pi_1^*, \pi_2^*) \frac{\partial}{\partial \ell_1} \pi_2^* = 0. \end{aligned}$$

This is a linear system in $\frac{\partial}{\partial \ell_1} \pi_1^*$ and $\frac{\partial}{\partial \ell_1} \pi_2^*$. After evaluating its solution at $\ell_1 = \ell$, and noticing that $\pi_1^*(\ell) = \pi_2^*(\ell)$, we get that $\frac{\partial}{\partial \ell_1} \pi_1^*|_{\ell_1 = \ell} = -\frac{\partial}{\partial \ell_1} \pi_2^*|_{\ell_1 = \ell}$.

Consider now TC^* as a function of (ℓ_1, π_1, π_2). Recall that $\frac{\partial}{\partial \pi_1} TC^*|_{\ell_1 = \ell, \pi_1 = \pi_1^*(\ell), \pi_2 = \pi_2^*(\ell)} = 0$, $\frac{\partial}{\partial \pi_2} TC^*|_{\ell_1 = \ell, \pi_1 = \pi_1^*(\ell), \pi_2 = \pi_2^*(\ell)} = 0$. Furthermore, it can be easily verified that $\frac{\partial^2}{\partial \ell_1^2} TC^*|_{\ell_1 = \ell, \pi_1 = \pi_1^*(\ell), \pi_2 = \pi_2^*(\ell)} =$

0. It follows that, after evaluating at $\ell_1 = \ell, \pi_1 = \pi_1^*(\ell), \pi_2 = \pi_2^*(\ell)$,

$$\begin{aligned}
\frac{d^2}{d\ell_1^2}TC^* &= 2\frac{\partial^2}{\partial\ell_1\partial\pi_1}TC^*\frac{\partial}{\partial\ell_1}\pi_1^* + 2\frac{\partial^2}{\partial\ell_1\partial\pi_2}TC^*\frac{\partial}{\partial\ell_1}\pi_2^* \\
&+ \frac{\partial^2}{\partial\pi_1^2}TC^*\left(\frac{\partial}{\partial\ell_1}\pi_1^*\right)^2 + \frac{\partial^2}{\partial\pi_2^2}TC^*\left(\frac{\partial}{\partial\ell_1}\pi_2^*\right)^2 + 2\frac{\partial^2}{\partial\pi_1\partial\pi_2}TC^*\frac{\partial}{\partial\ell_1}\pi_1^*\frac{\partial}{\partial\ell_1}\pi_2^* \\
&= 2\left(\frac{\partial^2}{\partial\ell_1\partial\pi_1}TC^* - \frac{\partial^2}{\partial\ell_1\partial\pi_2}TC^*\right)\frac{\partial}{\partial\ell_1}\pi_1^* \\
&+ \left(\frac{\partial^2}{\partial\pi_1^2}TC^* + \frac{\partial^2}{\partial\pi_2^2}TC^* - 2\frac{\partial^2}{\partial\pi_1\partial\pi_2}TC^*\right)\left(\frac{\partial}{\partial\ell_1}\pi_1^*\right)^2.
\end{aligned}$$

From Theorem 4.4 we get that $\frac{\partial}{\partial\ell_1}\pi_1^* < 0$. Hence, we need to prove that

$$2\left(\frac{\partial^2}{\partial\ell_1\partial\pi_1}TC^* - \frac{\partial^2}{\partial\ell_1\partial\pi_2}TC^*\right) + \left(\frac{\partial^2}{\partial\pi_1^2}TC^* + \frac{\partial^2}{\partial\pi_2^2}TC^* - 2\frac{\partial^2}{\partial\pi_1\partial\pi_2}TC^*\right)\left(\frac{\partial}{\partial\ell_1}\pi_1^*\right) > 0.$$

Rewrite TC^* as $P+M = 2P-S$, where P is the potential function, S the sum of idiosyncratic terms in the potential funtion and M the mixed term. Since $\frac{\partial}{\partial\ell_1}\pi_1^*$ solves equations B.2, the expression simplifies as

$$2\left(\frac{\partial^2}{\partial\ell_1\partial\pi_1}M - \frac{\partial^2}{\partial\ell_1\partial\pi_2}M\right) - \left(\frac{\partial^2}{\partial\pi_1^2}S + \frac{\partial^2}{\partial\pi_2^2}S\right)\left(\frac{\partial}{\partial\ell_1}\pi_1^*\right).$$

It is enough now to show that both terms are positive. $\frac{\partial^2}{\partial\pi_1^2}S > 0$ and $\frac{\partial^2}{\partial\pi_2^2}S > 0$ because the Nash equilibrium minimizes the idiosyncratic terms of the potential function. For $\ell_1 = \ell, \pi_1 = \pi_1^*(\ell), \pi_2 = \pi_2^*(\ell)$ we have

$$\frac{\partial^2}{\partial\ell_1\partial\pi_1}M - \frac{\partial^2}{\partial\ell_1\partial\pi_2}M = -2w_1^2((\pi_1^*)^2\gamma_1 + (1-\pi_1^*)^2\gamma_2)E[(Z_1 - Z_2)1_{\{\ell+\pi_1^*Z_1+(1-\pi_1^*)Z_2\leq 0\}}].$$

Hence, we are left to show that the expectation is negative. This follows immediately from the fact that $\pi_1^* \geq \frac{1}{2}$. □

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