Three Infinities in Mathematics:
A Comparative View

Tuesday, Thursday 2:30-3:45

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Course Description The idea of infinity is an idea that has been present in philosophical discussions for several centuries. Mathematicians, however, have attempted to give a precise meaning to this concept, and have done so in several different contexts. The aim of the course is to discuss various specific ways in which mathematicians introduced and discussed the notion of infinity. Specifically, this happened in three different branches of mathematics:

- Projective geometry (introduced formally at the beginning of the seventeenth century by Desargues);
- Calculus (introduced independently by Leibniz and Newton in the seventeenth century);
- Set theory (introduced by Cantor and Dedekind at the beginning of the twentieth century).

We will consider each one of these developments both with respect to their history, as well as to their current stage of development. The main intent of the course is to offer the students a precise idea of how infinity is considered and treated in these branches of mathematics, but also, and especially, to allow the student to compare these different approaches.

We will consider the following difficult question: Is it possible to say that different ideas of infinity occur in these three examples, or is it the case that there is only one notion of infinity, and what we are seeing are three different forms of this concept.

Let us try to clarify the question. As the student will realize during the course, it is clear that in each one of the examples there are different notions of infinity. In set theory, for example, there is a difference between cardinal and ordinal infinite numbers. In calculus one can distinguish between different orders of infinity and infinitesimals. Finally, in projective geometry one can differentiate between points, lines, planes at infinity.

We accept immediately this plurality of the notion of infinity; what we ask is whether the different aspects we are looking at reflect a unique underlying notion of infinity, or whether they point to intrinsically different notions.

A few words on each of the three components of the course.

The birth of projective geometry was inspired by the realization, among the painters of the Renaissance, that in order to offer realistic descriptions of reality, the painter had to include in his painting points and lines that did not exist in the real world. The best example is the horizon, which is an artificial line that in fact does not correspond to any actual line in the world one is considering. Similarly, when painting two parallel lines, they will be drawn in such a way as to meet at some point, while we know that that point does not exist in the space we are representing. These ideas can be made precise by replacing

the customary lines, planes, and spaces, with new objects (projective lines, planes, and spaces) that are obtained by "adding" points, lines, and planes to the traditional objects. Thus, infinity appears in geometry, transforming radically the way in which geometry is to be intended, and pretty soon acquires an analytic aspect as well, with the introduction of projective coordinates. In our class, we will show why the introduction of these new objects is important, and the consequences that they impose on geometry. We will then conclude by demonstrating how to introduce projective analytic geometry to give a very concrete sense to infinity in geometry.

The discussion on infinity in calculus will begin by distinguishing actual from potential infinity, with reference to Greek mathematics and philosophy. We will then outline on history of the origins and first developments of the calculus, both under its differential (Leibnitian) and fluxional (Newtonian) form. Passing from a survey of 18th-century algebraic analysis, we will then describe the program of arithmetization of analysis, and the birth of complex analysis, to finally come to the way the foundation of the calculus is today understood, both in standard and in the different forms of non-standard analysis.

The final part of the course concerns set theory. We will begin by summarily reconstruct its origins in Dedekind's foundational works devoted to the definition of natural and real numbers, and in Cantor's works on trigonometric series (in this way, incidentally, we will encounter the origins of set theory in a problem that arises naturally from the analysis that calculus gave birth to). We will then consider some of the well known paradoxes that appear when one attempts a first, naïve, codifications of the notion of set. To answer these paradoxes we will describe Russell's type theory and Zermelo's first axiomatization of a theory of sets. This will easily bring us to ZFC in its current form.

Course Learning Outcomes. In addition of extending the mathematical culture and knowledge of students, the course also aims at increasing their capacity of abstract and critical thinking, and their sense of historicity and evolution of mathematics, and, more generally, of science and knowledge. As part of the class, the students will learn the following mathematical techniques and ideas:

- The projective line, plane, and space
- Homogeneous coordinates for analytic geometry in the projective setting
- Projective invariants for the projective line (cross-ratio of points and lines)
- Conics: affine and projective treatment
- Fundamental techniques in calculus such as the calculation of tangents, derivatives and integrals in one variable.
- The fundamental theorem of calculus.
- Elements of classical mechanics, as they emerge in Newton's work and they evolve in an analytical setting (Euler and Lagrange).
- Elements of ordinary differential equations.
- Cardinal and ordinal numbers
- Axiom of choice
- Well-ordering

Instructional strategies. The instructors will begin any lecture with a series of questions students may have on what we discussed or read during the previous class. This should set the stage for the next topic. If students don't have questions, the instructors will ask questions, and we should all strive for a setting where we can openly discuss ideas and engage each other in the learning process. If the instructor will assign homework, they will also be

discussed the following week. After that, the instructor will offer the lecture, being always open to requirements of clarifications, questions, and discussion.

Prerequisites. Though involving some technical issues, the class does not require any specific prerequisite, though students are expected to be able to follow simple logical arguments, and be able to perform standard algebraic calculations. What is absolutely necessary is the willingness to look at apparently obvious questions, and to have an appreciation for the intermingling of logical and historical development of ideas.

Textbook. The teachers will provide several documents drawn from original and secondary sources and require student to work on them and comment them, in some cases. The following texts will provide necessary background for many of the classroom discussions.

C. H. Edwards, Jr., *The Historical Development of the Calculus*, Springer Verlag, New York, Berlin, etc., 1979.
G.Gentili, L.Simonutti, D.C.Struppa, *The birth of projective geometry in renaissance painting*, manuscript under publication (a copy will be provided).
A.Fraenkel, *Set Theory and Logic*, Addison-Wesley, Boston, 1966.

Grading. The teachers will regularly assign homework (ungraded, but object of discussion in class) and there will be both a mid-term and a final exam. The final grade will be computed as follows:

Class participation 30%
Mid-term 30%
Final exam 40%

The Chapman University Academic Integrity Policy. Chapman University is a community of scholars which emphasizes the mutual responsibility of all members to seek knowledge honestly and in good faith. Students are responsible for doing their own work, and academic dishonesty of any kind will not be tolerated anywhere in the university.

Students with Disabilities Policy. In compliance with ADA guidelines, students who have any condition, either permanent or temporary, that might affect their ability to perform in this class are encouraged to inform the instructor at the beginning of the term. The University, through the Center for Academic Success, will work with the appropriate faculty member who is asked to provide the accommodations for a student in determining what accommodations are suitable based on the documentation and the individual student needs. The granting of any accommodation will not be retroactive and cannot jeopardize the academic standards or integrity of the course.

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