Honors Course
From Natural to Real Numbers: a Conceptual Approach

Thursday, 7:00-9:50 pm

Instructor: Marco Panza

Phone #: 714-746-7235

Email: panzam10@gmail.com

Goal of the Course: To introduce some basic mathematical notions, in particular those of natural, rational and real numbers (plus some connected algebraic and analytical notions, like those of group, ring, field, convergence, density and continuity), by showing how they origin both in our effort to understand the real world and in our desire to get a precise and abstract rendering of some tools used to this purpose. On the one side, this should allow people unfamiliar with these notions to grasp them and to become able to work with them, on a justified basis, on the other side, this should also make people already familiar with them, or with some of them, aware of the reasons which made them have the features they actually have in modern mathematics (some historical consideration on the way these notions evolved will also be presented). The course will start by presenting an empirical theory of integer positive numbers, where these are defined as (types of) collections of strokes any of which is associated with distinct but equinoumerous collections of objects of whatever sort. This would allow to introduce the idea of an equivalence class (of these collections), and to develop finite arithmetic as a tool for counting and classifying collections. Reflecting on this theory will allow us to introduce a formal, axiomatic version of infinitary arithmetic and to define implicitly natural numbers as the objects this axiomatic system is supposed to deal with, giving particular attention to mathematical induction (as a basic method to prove infinitary theorems on these numbers), and to the notion of an arithmetical and a geometrical succession. We'll then move on to fractional numbers, understood as tools for studying the partition of given objects, and develop, firstly, an empirical theory of these numbers, and, secondly, a formal axiomatic theory of them, got by extension of the theory of real numbers. Reflecting on the passage from this latter theory to the former will allow us to introduce some basic algebraic notions as those of group, ring and field. Finally, we'll move on to real numbers, by introducing them, first, as tools for performing measurements of (empirical or geometrical) magnitudes and, then, as entities implicitly defined by a new extension of the theory of rational numbers, both trough Dedekind's cuts and Cauchy series (following Cantor approach). This should finally allow us to introduce the notions of density and continuity and to grasp their essential difference. Finally we will close by giving an informal presentation of Cantors's proofs of countability of the set of rational numbers and uncountability of that of real ones.

We will discuss the following topics:

1. Some fundamental questions: What is a number? What is an integer, and a fractionary number? What is a natural, a rational and a real number?
2. How did the notions of these numbers appear, both historically and anthropologically? In particular, what practical needs required their introduction?
3. How did these notions evolve and how were they rendered within a formal framework?

4. What is the essential difference between a tool allowing to answer practical questions and solving empirical problems, though general and abstract, and a formal theory which renders our practice of working with such a tool?
5. How and why did the need of the latter appear?
6. Does (basic) mathematics have a practical or empirical motivation and/or justification?
7. Does answering this question positively entail that (basic) mathematics is an empirical science, just as physics?
8. If not, why not?
9. What is continuity and why making this notion precise requires formalization?

Grading:

- 20% will be based on homework, which I will assign more or less regularly; in order for homework to be graded, they need to be done on time. No late assignments will be accepted.

- 40% will be a midterm exam designed to ascertain understanding of the basic notions which are being developed in class. The mid-term will be Thursday, November 1st. Only medical excuses will be accepted for make-up requests.

- 40% will be a final essay due at the day of the final exam.

- There will not be a final exam.

Textbook:

The instructor wrote a book in French on the topic of the course (including historical notes and philosophical remarks): M. Panza, *Nombres. Eléments de mathématiques pour philosophes*, ENS édition, Lyon, 2007. Some passages of this book will be translated (possibly with the help of students themselves) and used as a base for some lectures. Some passages from original works by Euclid, Dedekind, Frege and Cantor will also be used. Other lectures could include: R. Courant, H. Robbins, *What is Mathematics*, Oxford, 1941, (second edition, revised by I. Steward, Oxford, 1996; T.~Dantzig, *Numbers. The Language of Sciences. A Critical Survey Written for the Cultured Non-Mathematicians*, 3th ed.; London 1947, D.E. Littlewood, *The Skeleton Key of Mathematics*, London, 1949; M. Kline, *Mathematics for Liberal Arts*, Reading, Mass., 1967; I. Steward, *Concepts of Mathematics*, Harmondsworth (Middlesex), 1975; D. M. Davis, *The Nature and Power of Mathematics*, Princeton, 1993; W. Rudin, *Principles of Mathematical Analysis*, New York, etc, 3th ed., 1976 (this book can be find online here: https://www.scribd.com/doc/9654478/Principles-of-Mathematical-Analysis-Third-Edition-Walter-Rudin); T. M. Apostol, *Mathematical Analyis*, Reading (Mass.), 2nd ed., 1974 (this book can be find online here: http://webpages.iust.ac.ir/amtehrani/files/Addison%20Wesley%20-%20Mathematical%20Analysis%20_%20Apostol%20%285Th%20Ed%29%20%281981%29.pdf). Only some parts of the two last volumes will be used. These readings will enrich my lectures, and the discussion, and exercises in class.

Classroom format:

I will begin the class with a series of questions students may have on what we discussed or read during the previous class. This should set the stage for the next topic. If students don't have questions, I will ask questions, and we should all strive for a setting where we can openly discuss ideas and engage each other in the learning process. If I assign homeworks, they will also be discussed the following week.

After that, I'll offer a lecture, being always open to requirements of clarifications, questions, and discussion.

Then, I'll require some students to summarize the essential points I raised.

Finally, I'll try to close with a few minutes to reflect on what we have done, and to allow additional questions and thoughts to be discussed.